



# Polynomial Identities (Part 2)

Let's explore some other identities.

## 12.1 Revisiting an Old Theorem

Instructions to make a right triangle:

- Choose two integers.
- Make one side length equal to the sum of the squares of the two integers.
- Make one side length equal to the difference of the squares of the two integers.
- Make one side length equal to twice the product of the two integers.

Follow these instructions to make a few different triangles. Do you think the instructions always produce a right triangle? Be prepared to explain your reasoning.

## 12.2 Theorems and Identities

Here are the instructions to make a right triangle from earlier:

- Choose two integers.
  - Make one side length equal to the sum of the squares of the two integers.
  - Make one side length equal to the difference of the squares of the two integers.
  - Make one side length equal to twice the product of the two integers.
1. Using  $a$  and  $b$  for the two integers, write expressions for the three side lengths.

2. Why do these instructions make a right triangle when  $a \neq b$ ?

## 12.3

## Identifying Identities

Here is a list of equations. Circle all the equations that are identities. Be prepared to explain your reasoning.

1.  $a = -a$

2.  $a^2 + 2ab + b^2 = (a + b)^2$

3.  $a^2 - 2ab + b^2 = (a - b)^2$

4.  $a^2 - b^2 = (a - b)(a - b)$

5.  $(a + b)(a^2 - ab + b^2) = a^3 - b^3$

6.  $(a - b)^3 = a^3 - b^3 - 3ab(a + b)$

7.  $a^2(a - b)^4 - b^2(a - b)^4 = (a - b)^5(a + b)$



## 12.4 Egyptian Fractions



In Ancient Egypt, all non-unit fractions were represented as a sum of distinct unit fractions. For example,  $\frac{4}{9}$  would have been written as  $\frac{1}{3} + \frac{1}{9}$  (and not as  $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$  or any other form with the same unit fraction used more than once). Let's look at some different ways we can rewrite  $\frac{2}{15}$  as the sum of distinct unit fractions.

1. Use the formula  $\frac{2}{d} = \frac{1}{d} + \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}$  to rewrite the fraction  $\frac{2}{15}$ , then show that this formula is an identity.
2. Another way to rewrite fractions of the form  $\frac{2}{d}$  is given by the identity  $\frac{2}{d} = \frac{1}{d} + \frac{1}{d+1} + \frac{1}{d(d+1)}$ . Use it to rewrite the fraction  $\frac{2}{15}$ , then show that it is an identity.

### Are you ready for more?

For fractions of the form  $\frac{2}{pq}$ , that is, fractions with a denominator that is the product of two positive integers, the following formula can also be used:  $\frac{2}{pq} = \frac{1}{pr} + \frac{1}{qr}$ , where  $r = \frac{p+q}{2}$ . Use it to rewrite the fraction  $\frac{2}{45}$ , then show that it is an identity.

## Lesson 12 Summary

Sometimes we can think something is an identity when it actually isn't. Consider the following equations that are sometimes mistaken as identities:

$$(a + b)^2 = a^2 + b^2$$

$$(a - b)^2 = a^2 - b^2$$

Both of these are true for some very specific values of  $a$  and  $b$  (for example, when either  $a$  or  $b$  is 0), but they are not true for most values of  $a$  and  $b$ , for example  $a = 2$  and  $b = 1$  (try it!). The actual identities associated with the expressions on the left side are  $(a + b)^2 = a^2 + 2ab + b^2$  and  $(a - b)^2 = a^2 - 2ab + b^2$ .

Are polynomials the only types of expressions you can find in identities? Not at all! Here is an identity that shows a relationship between rational expressions:

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

We can show that this identity is true by adding the terms in the expression on the right using a common denominator:

$$\begin{aligned}\frac{1}{x+1} + \frac{1}{x(x+1)} &= \frac{1}{x+1} \cdot \frac{x}{x} + \frac{1}{x(x+1)} \\ &= \frac{x}{x(x+1)} + \frac{1}{x(x+1)} \\ &= \frac{x+1}{x(x+1)} \\ &= \frac{1}{x}\end{aligned}$$

An important difference from polynomial identities is that identities involving rational expressions could have a few exceptional values of  $x$  for which they are not true because the rational expressions on one side or the other are not defined. For example, the identity above is true for all values of  $x$  except  $x = 0$  and  $x = -1$ .