



# Sums and Products of Rational and Irrational Numbers

Let's make convincing arguments about why the sums and products of rational numbers and irrational numbers are always certain kinds of numbers.

## 21.1 Operations on Integers

Here are some examples of integers:

-25   -10   -2   -1   0   5   9   40

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
  - a. add up to another integer.
  - b. add up to a number that is *not* an integer.
2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
  - a. multiply to make another integer.
  - b. multiply to make a number that is *not* an integer.

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.

a.  $4 + 0.175 = 4.175$

b.  $\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10}$

c.  $-0.75 + \frac{14}{8} = \frac{-6}{8} + \frac{14}{8} = \frac{8}{8} = 1$

d.  $a$  is an integer:  $\frac{2}{3} + \frac{a}{15} = \frac{10}{15} + \frac{a}{15} = \frac{10+a}{15}$

2. Here is a way to explain why the sum of two rational numbers is rational:

Suppose  $\frac{a}{b}$  and  $\frac{c}{d}$  are fractions. That means that  $a, b, c$ , and  $d$  are integers, and  $b$  and  $d$  are not 0.

- a. Find the sum of  $\frac{a}{b}$  and  $\frac{c}{d}$ . Show your reasoning.

- b. In the sum, are the numerator and the denominator integers? How do you know?

- c. Use your responses to explain why the sum of  $\frac{a}{b} + \frac{c}{d}$  is a rational number.

3. Use the same reasoning as in the previous question to explain why the product of two rational numbers,  $\frac{a}{b} \cdot \frac{c}{d}$ , must be rational.

## Are you ready for more?

Consider numbers that are of the form  $a + b\sqrt{5}$ , where  $a$  and  $b$  are integers. Let's call such numbers *quintegers*.

Here are some examples of quintegers:

- $3 + 4\sqrt{5}$  ( $a = 3, b = 4$ )
- $-5 + \sqrt{5}$  ( $a = -5, b = 1$ )
- $7 - 2\sqrt{5}$  ( $a = 7, b = -2$ )
- $3$  ( $a = 3, b = 0$ ).

1. When we add two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose sum is not a quinteger.
  
2. When we multiply two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose product is not a quinteger.

## Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why  $\sqrt{2} + \frac{1}{9}$  is irrational.
  - Let  $s$  be the sum of  $\sqrt{2}$  and  $\frac{1}{9}$ , or  $s = \sqrt{2} + \frac{1}{9}$ .
  - Suppose  $s$  is rational.
  - a. Is  $s + -\frac{1}{9}$  rational or irrational? Explain how you know.
  - b. Evaluate  $s + -\frac{1}{9}$ . Is the sum rational or irrational?
  - c. Use your responses to explain why  $s$  cannot be a rational number, and therefore  $\sqrt{2} + \frac{1}{9}$  cannot be rational.
2. Use a similar reasoning as in the earlier question to explain why  $\sqrt{2} \cdot \frac{1}{9}$  is irrational. Here are some assumptions to get you started.
  - Let  $p$  be the product of  $\sqrt{2}$  and  $\frac{1}{9}$ , or  $p = \sqrt{2} \cdot \frac{1}{9}$ .
  - Suppose  $p$  is rational.

## 21.4

## Equations with Different Kinds of Solutions

1. Consider the equation  $4x^2 + bx + 9 = 0$ . Find a value of  $b$  so that the equation has:
  - a. 2 rational solutions
  - b. 2 irrational solutions
  - c. 1 solution
  - d. no solutions
2. Describe all the values of  $b$  that produce 2 solutions, 1 solution, and no solutions.
3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.
  - a. no solutions
  - b. 2 irrational solutions
  - c. 2 rational solutions
  - d. 1 solution



## Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to  $(x + 3)(x - 1) = 0$  are -3 and 1, which are rational. The solutions to  $x^2 - 8 = 0$  are  $\pm\sqrt{8}$ , which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example,  $\pm 4\sqrt{3}$  and  $1 + \sqrt{12}$ . What kind of numbers are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

- The sum of two rational numbers is rational. Here is one way to explain why it is true:
  - Any two rational numbers can be written  $\frac{a}{b}$  and  $\frac{c}{d}$ , where  $a, b, c$ , and  $d$  are integers, and  $b$  and  $d$  are not zero.
  - The sum of  $\frac{a}{b}$  and  $\frac{c}{d}$  is  $\frac{ad+bc}{bd}$ . The denominator is not zero because neither  $b$  nor  $d$  is zero.
  - Multiplying or adding two integers always gives an integer, so we know that  $ad, bc, bd$  and  $ad + bc$  are all integers.
  - If the numerator and denominator of  $\frac{ad+bc}{bd}$  are integers, then the number is a fraction, which is rational.
- The product of two rational numbers is rational. We can show why in a similar way:
  - For any two rational numbers  $\frac{a}{b}$  and  $\frac{c}{d}$ , where  $a, b, c$ , and  $d$  are integers, and  $b$  and  $d$  are not zero, the product is  $\frac{ac}{bd}$ .
  - Multiplying two integers always results in an integer, so both  $ac$  and  $bd$  are integers. Therefore,  $\frac{ac}{bd}$  is a rational number.

What about two irrational numbers?

- The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
  - $\sqrt{3}$  and  $-\sqrt{3}$  are both irrational, but their sum is 0, which is rational.
  - $\sqrt{3}$  and  $\sqrt{5}$  are both irrational, and their sum is irrational.
- The product of two irrational numbers could be either rational or irrational. We can show this through examples:
  - $\sqrt{2}$  and  $\sqrt{8}$  are both irrational, but their product is  $\sqrt{16}$ , or 4, which is rational.
  - $\sqrt{2}$  and  $\sqrt{7}$  are both irrational, and their product is  $\sqrt{14}$ , which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

- The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
  - Let  $R$  be a rational number and  $I$  an irrational number. We want to show that  $R + I$  is irrational.
  - Suppose  $s$  represents the sum of  $R$  and  $I$  ( $s = R + I$ ), and suppose  $s$  is rational.
  - If  $s$  is rational, then  $s + -R$  would also be rational, because the sum of two rational numbers is rational.
  - $s + -R$  is not rational, however, because  $(R + I) + -R = I$ .
  - $s + -R$  cannot be both rational and irrational, which means that our original assumption that  $s$  is rational was incorrect.  $s$ , which is the sum of a rational number and an irrational number, must be irrational.
- The product of a nonzero rational number and an irrational number is irrational. We can show why this is true in a similar way:
  - Let  $R$  be rational and  $I$  irrational. We want to show that  $R \cdot I$  is irrational.
  - Suppose  $p$  is the product of  $R$  and  $I$  ( $p = R \cdot I$ ), and suppose  $p$  is rational.
  - If  $p$  is rational, then  $p \cdot \frac{1}{R}$  would also be rational because the product of two rational numbers is rational.
  - $p \cdot \frac{1}{R}$  is not rational, however, because  $R \cdot I \cdot \frac{1}{R} = I$ .
  - $p \cdot \frac{1}{R}$  cannot be both rational and irrational, which means our original assumption that  $p$  is rational was false.  $p$ , which is the product of a rational number and an irrational number, must be irrational.