Polynomial Identities (Part 2)

Let's explore some other identities.

12.1 Revisiting an Old Theorem

Instructions to make a right triangle:

- Choose two integers.
- Make one side length equal to the sum of the squares of the two integers.
- Make one side length equal to the difference of the squares of the two integers.
- Make one side length equal to twice the product of the two integers.

Follow these instructions to make a few different triangles. Do you think the instructions always produce a right triangle? Be prepared to explain your reasoning.

12.2 Theorems and Identities

Here are the instructions to make a right triangle from earlier:

- Choose two integers.
- Make one side length equal to the sum of the squares of the two integers.
- Make one side length equal to the difference of the squares of the two integers.
- Make one side length equal to twice the product of the two integers.

1. Using a and b for the two integers, write expressions for the three side lengths.

2. Why do these instructions make a right triangle when $a \neq b$?

12.3 Identifying Identities

Here is a list of equations. Circle all the equations that are identities. Be prepared to explain your reasoning.

1. $a = \text{-}a$
2. $a^2 + 2ab + b^2 = (a + b)^2$
3. $a^2 - 2ab + b^2 = (a - b)^2$
4. $a^2 - b^2 = (a - b)(a - b)$
5. $(a + b)(a^2 - ab + b^2) = a^3 - b^3$
6. $(a - b)^3 = a^3 - b^3 - 3ab(a + b)$
7. $a^2(a - b)^4 - b^2(a - b)^4 = (a - b)^5(a + b)$

12.4 Egyptian Fractions

In Ancient Egypt, all non-unit fractions were represented as a sum of distinct unit fractions. For example, $\frac{4}{9}$ would have been written as $\frac{1}{3} + \frac{1}{9}$ (and not as $\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$ or any other form with the same unit fraction used more than once). Let’s look at some different ways we can rewrite $\frac{2}{15}$ as the sum of distinct unit fractions.

1. Use the formula $\frac{2}{d} = \frac{1}{d} + \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}$ to rewrite the fraction $\frac{2}{15}$, then show that this formula is an identity.

2. Another way to rewrite fractions of the form $\frac{2}{d}$ is given by the identity $\frac{2}{d} = \frac{1}{d} + \frac{1}{d+1} + \frac{1}{d(d+1)}$. Use it to rewrite the fraction $\frac{2}{15}$, then show that it is an identity.

Are you ready for more?

For fractions of the form $\frac{2}{pq}$, that is, fractions with a denominator that is the product of two positive integers, the following formula can also be used: $\frac{2}{pq} = \frac{1}{pr} + \frac{1}{qr}$, where $r = \frac{p+q}{2}$. Use it to rewrite the fraction $\frac{2}{45}$, then show that it is an identity.

Lesson 12 Summary

Sometimes we can think something is an identity when it actually isn't. Consider the following equations that are sometimes mistaken as identities:

$$(a+b)^2 = a^2 + b^2$$

$$(a-b)^2 = a^2 - b^2$$

Both of these are true for some very specific values of a and b (for example, when either a or b is 0), but they are not true for most values of a and b, for example $a = 2$ and $b = 1$ (try it!). The actual identities associated with the expressions on the left side are $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

Are polynomials the only types of expressions you can find in identities? Not at all! Here is an identity that shows a relationship between rational expressions:

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

We can show that this identity is true by adding the terms in the expression on the right using a common denominator:

$$\begin{aligned} \frac{1}{x+1} + \frac{1}{x(x+1)} &= \frac{1}{x+1} \cdot \frac{x}{x} + \frac{1}{x(x+1)} \\ &= \frac{x}{x(x+1)} + \frac{1}{x(x+1)} \\ &= \frac{x+1}{x(x+1)} \\ &= \frac{1}{x} \end{aligned}$$

An important difference from polynomial identities is that identities involving rational expressions could have a few exceptional values of x for which they are not true because the rational expressions on one side or the other are not defined. For example, the identity above is true for all values of x except $x = 0$ and $x = -1$.

