Conditions for Triangle Similarity

Goals

- Prove the Angle-Angle Triangle Similarity Theorem (in writing).

Learning Targets

- I can explain why the Angle-Angle Triangle Similarity Theorem works.

Lesson Narrative

In this lesson, students use the Angle-Side-Angle Triangle Congruence Theorem as the basis for proving the Angle-Angle Triangle Similarity Theorem. Students make use of the structure of the Angle-Side-Angle Triangle Congruence Theorem along with their understanding of dilations to prove that two triangles with 2 pairs of corresponding congruent angles must be similar (MP7).

Students then make use of the Triangle Sum Theorem along with the Angle-Angle Triangle Similarity Theorem to determine other angle conditions that could be used to determine whether two triangles are similar.

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. We recommend making technology available.

Standards

Building On	**HSG-CO.B.8, HSG-CO.C.10, HSG-SRT.A.2**
Addressing	**HSG-SRT.A.3**
Building Toward	**HSG-SRT.A.3**

Instructional Routines

- Draw It
- Math Talk
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports

Required Materials

Materials to Gather

- Geometry toolkits (HS): Activity 2
- Protractors: Activity 2
- Rulers marked with centimeters: Activity 2

Required Preparation

Activity 2:

Students will continue adding to their reference chart in this activity. Be prepared to add to the class display. The Blank Reference Chart for students and a teacher copy of a completed version are available in the blackline masters for the unit.

If there are multiple sections of this course in the same classroom, consider hiding entries on the class reference chart and revealing them at the appropriate time rather than making multiple displays.

Student Facing Learning Goals

Let's prove that some triangles are similar.

9.1 Math Talk: Angle-Side-Angle as a Helpful Tool

10 min

Warm-up

Activity Narrative

This *Math Talk* focuses on using the idea of congruence to show that two triangles are similar. It encourages students to think about the definitions of congruence and similarity and to rely on what they know about triangle congruence to mentally solve problems. The strategies elicited here will be helpful later in the lesson when students examine triangle similarity theorems.

To show congruence between triangles, students need to look for and make use of structure (MP7).

Standards

Building On	**HSG-CO.B.8**
Building Toward	**HSG-SRT.A.3**

Instructional Routines

- Math Talk
- MLR8: Discussion Supports

Launch

Tell students to close their books or devices (or to keep them closed). Reveal one problem at a time. For each problem:

- Give students quiet think time, and ask them to give a signal when they have an answer and a strategy.
- Invite students to share their strategies, and record and display their responses for all to see.
- Use the questions in the *Activity Synthesis* to involve more students in the conversation, before moving to the next problem.

Keep all previous problems and work displayed throughout the talk.

Access for Students with Disabilities

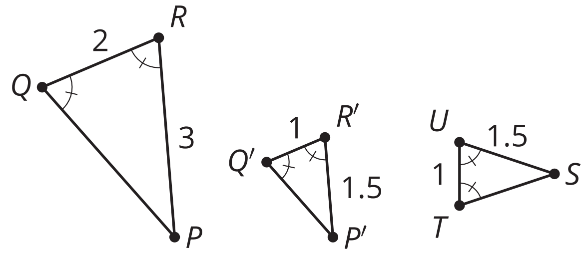
Action and Expression: Internalize Executive Functions. To support working memory, provide students with sticky notes or mini whiteboards.

Supports accessibility for: Memory, Organization

Student Task Statement

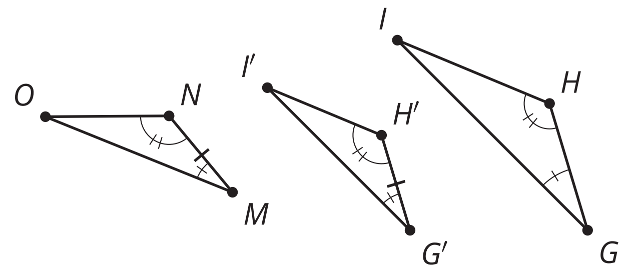
Justify each statement.

$\angle Q \cong \angle R \cong \angle Q' \cong \angle R' \cong \angle T \cong \angle U$



- Triangle $P'Q'R'$ is congruent to triangle STU.
- Triangle PQR is similar to triangle STU.

$\angle G \cong \angle G' \cong \angle M, \angle H \cong \angle H' \cong \angle N$



- Triangle $G'H'I'$ is congruent to triangle MNO.
- Triangle GHI is similar to triangle MNO.

Student Response

Sample responses:

$\triangle P'Q'R' \cong \triangle STU$

- Translate and rotate to take triangle $P'Q'R'$ onto triangle STU.
  Use the Angle-Side-Angle Triangle Congruence Theorem.
  Use the Side-Angle-Side Triangle Congruence Theorem.

$\triangle PQR \sim \triangle STU$

- By dilating triangle PQR, using center P, by a scale factor of $\frac{1}{2}$, and then translate and rotate to take triangle $P'Q'R'$ onto triangle STU.
  Dilate triangle PQR, using center P, by a scale factor of $\frac{1}{2}$, and then use that $Q'R'$ is the same length as TU, so $P'Q'R'$ is congruent to STU by the Angle-Side-Angle Triangle Congruence Theorem. Congruent figures can be taken onto each other using rigid motions, so there is a dilation and sequence of rigid motions that takes PQR onto STU.

$\triangle G'H'I' \cong \triangle MNO$

- Translate and rotate to take triangle $G'H'I'$ onto triangle MNO.
  Use the Angle-Side-Angle Triangle Congruence Theorem.

$\triangle GHI \sim \triangle MNO$

- Dilate triangle GHI by a scale factor of $\frac{MN}{GH}$. Then $G'H'$ is the same length as MN, so $G'H'I'$ is congruent to MNO by the Angle-Side-Angle Triangle Congruence Theorem. $G'H'I'$ is a dilation of GHI, so there is a dilation and sequence of rigid motions that takes GHI onto MNO. Therefore, GHI is similar to MNO.

Activity Synthesis

The goal of this discussion is to review students' strategies for finding similarity and congruence between triangles.

To involve more students in the conversation, consider asking:

- "Who can restate ________'s reasoning in a different way?"
- "Did anyone use the same strategy but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to ________'s strategy?"
- "Do you agree or disagree? Why?"
- "What connections to previous problems do you see?"

If no students bring up the Angle-Side-Angle Triangle Congruence Theorem, ask students how they might use the triangle congruence theorems that they know. Then ask students, "If a dilation takes triangle ABC to triangle $A'B'C'$, and triangle $A'B'C'$ is congruent to triangle DEF, how do we know that triangle ABC is similar to triangle DEF?" (After the dilation, the figures are congruent which means there is a sequence of rigid motions that takes $A'B'C'$ onto triangle DEF. The same sequence of rigid motions following the dilation to make triangle $A'B'C'$ shows that triangle ABC and triangle DEF are similar by the definition of similarity.)

Access for English Language Learners

MLR8 Discussion Supports. Display sentence frames to support students when they explain their strategy. Examples:

- "First, I ________ because ________."
- "I noticed ________, so I ________."

Advances: Speaking, Representing

9.2 How Many Pieces?

15 min

Activity Narrative

While the Angle-Angle Triangle Similarity Theorem follows logically from the Angle-Side-Angle Triangle Congruence Theorem (dilate one triangle by a scale factor determined by the ratio of the sides between the known corresponding angle pairs, and the dilated triangle will be congruent to the target by the Angle-Side-Angle Triangle Congruence Theorem, so the original triangle must be similar to the target), it can still be unintuitive.

Students experienced the Angle-Angle Triangle Similarity Theorem briefly in a previous unit when they looked at examples of triangles constructed so that their angles were the same but their side lengths were undetermined. In middle school, students confirmed the Angle-Angle Triangle Similarity Theorem informally. However, their proofs until this point have tended to rely on knowing something about the proportionality of side lengths.

Students might be unsure that the same scale factor will work for all three pairs of corresponding sides when all they know is the measure of two angles of the triangle. It is a surprising result when you put it that way. This activity starts by building students' intuition about the Angle-Angle Triangle Similarity Theorem, to motivate the proof of the theorem.

Making dynamic geometry software available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Building On **HSG-CO.B.8, HSG-SRT.A.2**

Addressing **HSG-SRT.A.3**

Instructional Routines

- Draw It

Launch

Arrange students in groups of 4. Give students 5–7 minutes to work quietly to draw their triangles. Then give students 5 minutes to compare their triangles and prove their conjectures. Follow with a whole-class discussion.

Give students a few minutes of work time to draw the triangles. If multiple students are struggling, pause for a brief whole-class discussion. Invite a student to demonstrate a technique for drawing a triangle with one 45-degree angle and one 30-degree angle. Remind students that they can choose whatever side lengths they want, if that information isn't specified.

Access for Students with Disabilities

Action and Expression: Provide Access for Physical Action. Provide access to tools and assistive technologies, such as dynamic geometry software.
Supports accessibility for: Visual-Spatial Processing, Conceptual Processing, Organization

Student Task Statement

For each problem, draw 2 triangles that have the listed properties. Try to make them as different as possible.

1. One angle is 45 degrees.
2. One angle is 45 degrees, and another angle is 30 degrees.
3. One angle is 45 degrees, and another angle is 30 degrees. The lengths of a pair of corresponding sides are 2 cm and 6 cm.
4. Compare your triangles with the other triangles from your group. Do any of the conditions guarantee that the triangles will be similar? Make a conjecture.
5. Prove your conjecture.

Student Response

1. Answers vary. (There are many triangles with a 45-degree angle.)
2. Answers vary. (All the triangles will be similar but could be any size.)
3. Answers vary. (All the triangles will be similar. There are only a few possibilities here.)
4. Two pairs of corresponding angles being congruent guarantee that the triangles are similar.
5. Two triangles are similar when there is a dilation that takes one of the triangles to an image that is congruent to the other figure. Label the first triangle ABC and the second triangle PQR so that angle A is congruent to angle P and angle B is congruent to angle Q. Dilate triangle ABC by the scale factor $\frac{PQ}{AB}$. Then the length of $A'B'$ will be the same as the length of PQ, and $A'B'$ will be congruent to PQ. We already know that angle A is congruent to angle P and that angle B is congruent to angle Q, so we have enough information to say that $A'B'C'$ is congruent to PQR by the Angle-Side-Angle Triangle Congruence Theorem. Because $A'B'C'$ is a dilation of ABC, that means there is a dilation and sequence of rigid motions that takes ABC onto PQR, which means that the two triangles are similar, even though all we knew was that two pairs of corresponding angles were congruent.

Building on Student Thinking

If students struggle with the proof of their conjecture, invite them to go back to what they discussed during the *Warm-up*. Here are some questions that can help progress the reasoning.

- "Is there a dilation you could use that would make those triangles congruent? How does that help you prove that they are similar?" (Yes, dilate the smaller triangle with a scale factor of 3. Then I can show there's a dilation and sequence of rigid motions that takes one triangle onto the other, so they're similar.)
- "How many sides did you need to have to figure out a dilation for the triangles with sides given? Why? Could you measure fewer sides?" (I only needed one side to figure out that I could scale the triangle by a scale factor of 3. I could measure just one pair of sides, and scale up by the right scale factor to make those sides match.)

Activity Synthesis

The goal of this discussion is for students to be convinced that the Angle-Angle Triangle Similarity Theorem is a valid way to prove that two triangles are similar.

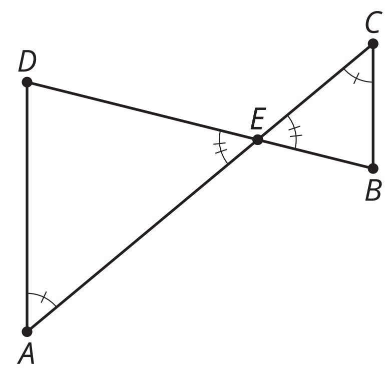
Invite students to summarize the main points of the proof:

- Choose the pair of corresponding sides between the known congruent angles to generate a scale factor.
- Dilate one triangle using that scale factor.
- The triangles must be congruent now based on the Angle-Side-Angle Triangle Congruence Theorem.
- The triangles can be taken onto each other by dilation and rigid motions. Therefore, they are similar.

Add the following theorem to the class reference chart, and ask students to add it to their reference charts:

Angle-Angle Triangle Similarity Theorem: In two triangles, if two pairs of corresponding angles are congruent, then the triangles must be similar. (Theorem)

$\angle A \cong \angle C, \angle DEA \cong \angle BEC$, **so** $\triangle DEA \sim \triangle BEC$



9.3 Any Two Angles?

10 min

Activity Narrative

This activity expands upon the Angle-Angle Triangle Similarity Theorem by requiring the use of the Triangle Angle Sum Theorem to find two pairs of congruent angles. Once similarity is established, students use the scale factor to determine missing side lengths on the triangles.

This is the first time Math Language Routine 5: *Co-Craft Questions* is suggested in this course. In this routine, students are given a context or situation, often in the form of a problem stem (for example, a story, image, video, or graph) with or without numerical values. Students develop mathematical questions that can be asked about the situation. A typical prompt is: "What mathematical questions could you ask about this situation?" The purpose of this routine is to allow students to make sense of a context before feeling pressure to produce answers, and to develop students' awareness of the language used in mathematics problems.

Access for English Language Learners

This activity uses the *Co-Craft Questions* math language routine to advance reading and writing as students make sense of a context and practice generating mathematical questions.

Standards

Building On	**HSG-CO.C.10**
Addressing	**HSG-SRT.A.3**

Instructional Routines

- MLR5: Co-Craft Questions

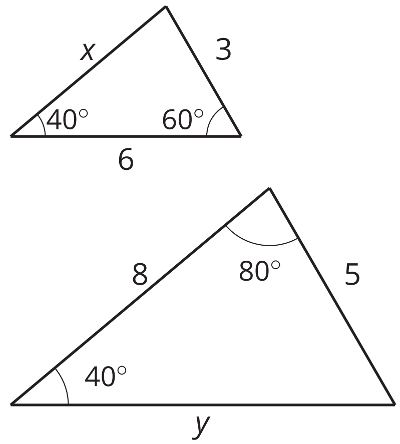
Launch

Arrange students in groups of 2. Use *Co-Craft Questions* to give students an opportunity to familiarize themselves with the context, and to practice producing the language of mathematical questions.

- Display only the problem stem and related image, without revealing the question.
- Ask students, "What mathematical questions could you ask about this situation?"
- Give students 1–2 minutes to write a list of mathematical questions that could be asked about the situation before comparing questions with a partner.
- As partners discuss, support students in using conversation and collaboration skills to generate and refine their questions, for instance, by revoicing a question, seeking clarity, or referring to their written notes. Listen for how students use language about the questions "What are the values of x and y?" "What are the measures of the missing angles?" and "Are the triangles similar?"
- Invite several groups to share one question with the class and record for all to see. Ask the class to make comparisons among the shared questions and their own. Ask, "What do these questions have in common? How are they different?" Listen for and amplify questions that focus on angles and similarity.
- Reveal the question "How can you show that the triangles are similar?" and give students 1–2 minutes to compare it to their own question and those of their classmates. Invite students to identify similarities and differences by asking:
    - "Which of your questions is most similar to or different from the one provided? Why?"
    - "How do your questions relate to similarity?"

Student Task Statement

Here are 2 triangles. One triangle has a 60 degree angle and a 40 degree angle. The other triangle has a 40 degree angle and an 80 degree angle.



1. How can you show that the triangles are similar?
2. How long are the sides labeled x and y?

Student Response

1. I know that the sum of the angles of a triangle is 180 degrees, sso I figured out the missing angle in each triangle. Both triangles have angles of 40, 60, and 80 degrees, so they must be similar by the Angle-Angle Triangle Similarity Theorem.
2. x is 4.8 units long, and y is 10 units long

Are You Ready for More?

Under what conditions is there an Angle-Angle Quadrilateral Similarity Theorem? What about an Angle-Angle-Angle Quadrilateral Similarity Theorem? Explain or show your reasoning.

Extension Student Response

Sample response: With 2, 3, or even 4 angles congruent, there is not enough to guarantee similarity. For example, rectangles have all right angles, but are not necessarily similar.

Activity Synthesis

The purpose of this discussion is for students to recognize that there are some cases in which knowing two pairs of angles in a pair of triangles is enough to show that they are similar even if there is only one pair of congruent corresponding angles.

Invite groups to share their reasoning about why the triangles are similar. Listen for students to share that the third angle of each triangle can be found based on the sum of the angles. If it is not mentioned, ask students to refer to their reference chart to find the name of the theorem that shows this idea (Triangle Angle Sum Theorem).

Ask students what is special about the given angles that lets us know that triangles are similar when only two angles are given for each triangle. (The triangles are similar when the angles given are either:

- Two pairs of congruent, corresponding angles or
- One pair of congruent, corresponding angles and the other angles are not corresponding, but the sum of the two non-corresponding angles along with one of the given corresponding angles is 180 degrees.)

It is ok if students cannot precisely articulate the second condition, as long as they understand the process of using the Triangle Angle Sum Theorem to find the missing angles and then show that all pairs of corresponding angles are congruent.

Lesson Synthesis

The main ideas to draw out of this lesson are:

- The Angle-Angle Triangle Similarity Theorem: If two pairs of corresponding angles in triangles are congruent, then the triangles are similar.
- Knowing the measures of any two angles from one triangle, and any two angles of the other triangle, is enough information to determine if the Angle-Angle Triangle Similarity Theorem can be used. The given angles do not have to be corresponding.
- If a dilation can take Figure A to an image that is congruent to Figure B, then Figure A and Figure B are similar.

Invite students to make up their own examples of pairs of triangles that are similar and pairs of triangles that are not similar. If students are familiar with the game, they could set up their examples as two truths and a lie. Ask students to trade with a partner and justify which triangles are similar.

9.4 Any Four Angles?

Cool-down

5 min

Standards

Addressing **HSG-SRT.A.3**

Student Task Statement

Priya noticed in the last activity that between the 2 triangles, you need to know only 4 angles to show that they are similar. She wondered which fourth angle would work to prove that triangle RST is similar to triangle EFG.

In triangle RST:

- Angle R is 90°
- Angle S is 25°
- Angle T is x°

In triangle EFG:

- Angle E is 90°
- Angle F is y°
- Angle G is z°

Draw a sketch of the triangles. Give an example of a value for one of x, y, or z that would guarantee that triangle RST is similar to triangle EFG. Explain your reasoning.

Student Response

Sample responses:

- x must be 65, but that is not enough to guarantee similarity with the other triangle if we do not know y or z.
- If $y = 25$, then the triangles are similar by the Angle-Angle Triangle Similarity Theorem.

- If $z = 65$, then the triangles are similar by the Triangle Angle Sum Theorem and the Angle-Angle Triangle Similarity Theorem.

Responding to Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

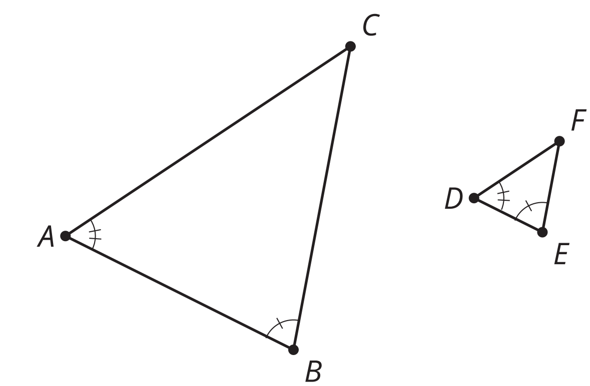
Lesson 9 Summary

We already know that when two figures are congruent there is a sequence of rigid motions that takes one figure onto the other. So, if a dilation takes Figure A to an image that is congruent to Figure B, then Figure A and Figure B are similar because there is a sequence of a dilation and rigid motions that takes Figure A onto Figure B.

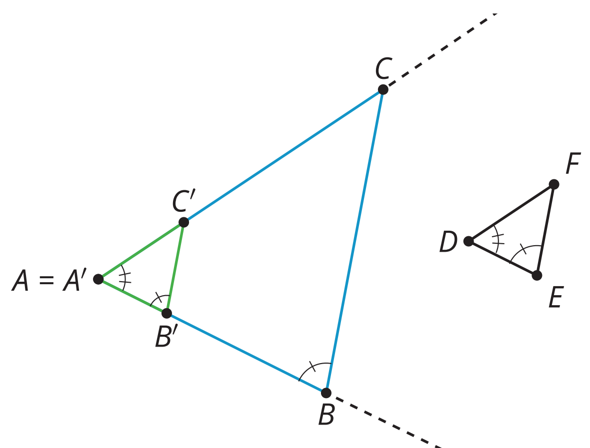
We can use this idea to show that when two angles of one triangle are congruent to two angles of a second triangle, then the two triangles are similar. We call this the Angle-Angle Triangle Similarity Theorem.

In the diagram, angle A is congruent to angle D, and angle B is congruent to angle E. If a sequence of rigid motions and dilations moves the first figure so that it fits exactly over the second, then we have shown that the Angle-Angle Triangle Similarity Theorem is true.

$\angle A \cong \angle D, \angle B \cong \angle E$



Dilate triangle ABC by the ratio $\frac{DE}{AB}$, so that $A'B'$ is congruent to DE. Now triangle $A'B'C'$ is congruent to triangle DEF by the Angle-Side-Angle Triangle Congruence Theorem, which means that there is a sequence of rotations, reflections, and translations that takes $A'B'C'$ onto DEF.

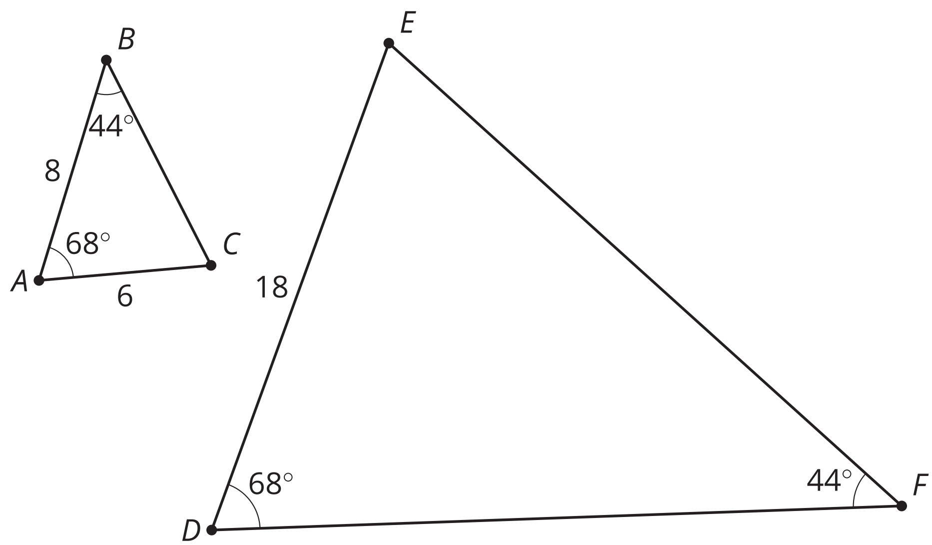


Therefore, a dilation followed by a sequence of rotations, reflections, and translations will take triangle ABC onto triangle DEF, which is the definition of similarity. We have shown that a dilation and a sequence of rigid motions takes triangle ABC to triangle DEF, so the triangles are similar.

Lesson 9 Practice Problems

1 Student Task Statement

What is the length of segment DF?



Solution

24 units

2 Student Task Statement

In triangle ABC, angle A is 35° and angle B is 20°. Select **all** triangles that are similar to triangle ABC.

A. triangle DEF where angle D is 35° and angle E is 20°

B. triangle GHI where angle G is 35° and angle I is 30°

C. triangle JKL where angle J is 35° and angle L is 125°

D. triangle MNO where angle N is 20° and angle O is 125°

E. triangle PQR where angle Q is 20° and angle R is 30°

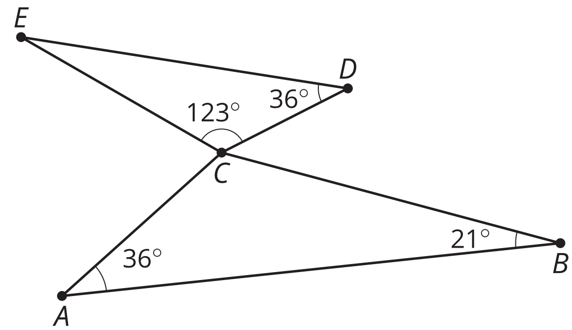
Solution

A, C, D

3

Student Task Statement

Decide whether triangles ABC and DEC are similar. Explain or show your reasoning.



Solution

Yes. Sample response: By the Triangle Angle Sum Theorem, there are two pairs of congruent corresponding angles, so they are similar by the Angle-Angle Triangle Similarity Theorem.

4

from Unit 2, Lesson 8

Student Task Statement

Lin is trying to convince Andre that all circles are similar. Help her write a valid justification for why all circles are similar.

Solution

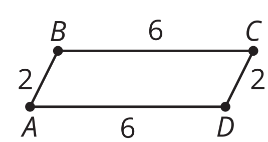
Sample response: I can translate the center of any circle to the center of another, and then dilate from that center by an appropriate scale factor, so they must be similar.

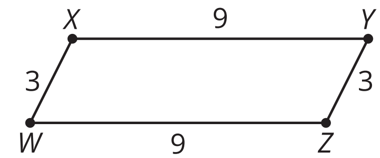
5

from Unit 2, Lesson 8

Student Task Statement

Must these parallelograms be similar? Explain your reasoning.





Solution

Sample response: No, the parallelograms might be similar. Because we do not know if the corresponding angles are congruent, we do not know for sure if the parallelograms are similar.

6 from Unit 2, Lesson 7

Student Task Statement

Determine if each statement must be true, could possibly be true, or definitely can't be true. Explain or show your reasoning.

a. An equilateral triangle and a right triangle are similar.

b. A right triangle and an isosceles triangle are similar.

Solution

Sample response:

a. Definitely can't be true. A right triangle has a 90° angle in it, and an equilateral triangle cannot have a 90° angle. They can't be similar because corresponding angles in similar figures must have the same measure.

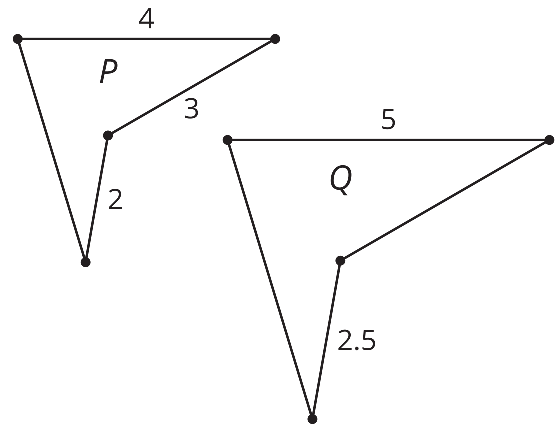
b. Could possibly be true. If the right triangle is isosceles—for example, a triangle with sides 1 unit, 1 unit, $\sqrt{2}$ units, and a triangle with sides 4 units, 4 units, $4\sqrt{2}$ units.

7 from Unit 2, Lesson 6

Student Task Statement

Quadrilaterals Q and P are similar.

What is the scale factor of the dilation that takes P to Q?



A. $\frac{3}{5}$

B. $\frac{4}{5}$

C. $\frac{5}{4}$

D. $\frac{5}{3}$

Solution

C

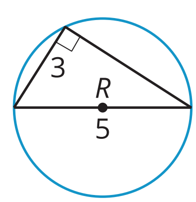
8 from Unit 2, Lesson 1

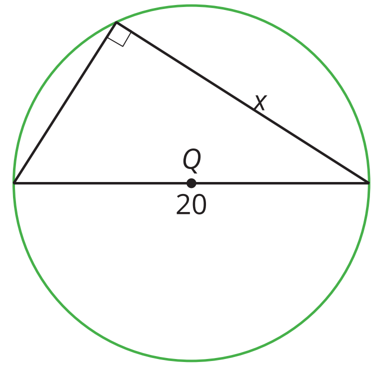
Student Task Statement

The circle centered at Q is a scaled copy of the circle centered at R.

a. Find the scale factor.

b. Find the value of x.





Solution

a. 4

b. 16

