Completing the Square (Part 3)

Let's complete the square for some more complicated expressions.

14.1 Perfect Squares in Two Forms

Previously, we saw that $(x+3)^2$ can be expanded to standard form as $x^2+2\cdot 3x+3^2$.

1. Expand $(5x+3)^2$ into standard form.

2. Be prepared to share a conjecture about the relationship between the coefficients 5 and 3 in the factored form and the values in standard form.

14.2 Perfect in a Different Way

1. Write each expression in standard form:

   a. $(4x + 1)^2$

   b. $(5x - 2)^2$

   c. $(\frac{1}{2}x + 7)^2$

   d. $(3x + n)^2$

   e. $(kx + m)^2$

2. Decide if each expression is a perfect square. If so, write an equivalent expression of the form $(kx + m)^2$. If not, suggest one change to turn it into a perfect square.

   a. $4x^2 + 12x + 9$

   b. $4x^2 + 8x + 25$

14.3 When All the Stars Align

1. Find the value of c to make each expression in the left column a perfect square in standard form. Then, write an equivalent expression in the form of squared factor. In the last row, write your own pair of equivalent expressions.

standard form $(ax^2 + bx + c)$	squared factor $(kx + m)^2$
$100x^2 + 80x + c$	
$36x^2 - 60x + c$	
$25x^2 + 40x + c$	
$0.25x^2 - 14x + c$	

2. Solve each equation by completing the square:

$25x^2 + 40x = -12$

$36x^2 - 60x + 10 = -6$

14.4 Putting Stars into Alignment

Here are three methods for solving $3x^2 + 8x + 5 = 0$.

Try to make sense of each method.

Method 1:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ (3x+5)(x+1) &= 0 \end{aligned}$$

$$x = -\frac{5}{3} \quad \text{or} \quad x = -1$$

Method 2:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ 9x^2 + 24x + 15 &= 0 \\ (3x)^2 + 8(3x) + 15 &= 0 \\ U^2 + 8U + 15 &= 0 \\ (U+5)(U+3) &= 0 \end{aligned}$$

$$\begin{aligned} U = -5 &\quad \text{or} \quad U = -3 \\ 3x = -5 &\quad \text{or} \quad 3x = -3 \\ x = -\frac{5}{3} &\quad \text{or} \quad x = -1 \end{aligned}$$

Method 3:

$$\begin{aligned} 3x^2 + 8x + 5 &= 0 \\ 9x^2 + 24x + 15 &= 0 \\ 9x^2 + 24x + 16 &= 1 \\ (3x+4)^2 &= 1 \end{aligned}$$

$$\begin{aligned} 3x + 4 = 1 &\quad \text{or} \quad 3x + 4 = -1 \\ x = -1 &\quad \text{or} \quad x = -\frac{5}{3} \end{aligned}$$

Once you understand the methods, use each method at least one time to solve these equations.

1. $5x^2 + 17x + 6 = 0$

2. $6x^2 + 19x = -10$

3. $8x^2 - 33x + 4 = 0$

4. $8x^2 - 26x = -21$

5. $10x^2 + 37x = 36$

6. $12x^2 + 20x - 77 = 0$

Are you ready for more?

Find the solutions to $3x^2 - 6x + \frac{9}{4} = 0$. Explain your reasoning.

Lesson 14 Summary

In earlier lessons, we worked with perfect squares such as $(x+1)^2$ and $(x-5)(x-5)$. We learned that their equivalent expressions in standard form follow a predictable pattern:

- In general, $(x+m)^2$ can be written as $x^2 + 2mx + m^2$.
- If a quadratic expression of the form $ax^2 + bx + c$ is a perfect square, and the value of a is 1, then the value of b is $2m$, and the value of c is m^2 for some value of m.

In this lesson, the variables in the factors being squared had coefficients other than 1, for example $(3x+1)^2$ and $(2x-5)(2x-5)$. Their equivalent expressions in standard form also followed the same pattern we saw earlier.

squared factor	standard form
$(3x+1)^2$	$(3x)^2 + 2(3x)(1) + 1^2$ or $9x^2 + 6x + 1$
$(2x-5)^2$	$(2x)^2 + 2(2x)(\text{-}5) + (\text{-}5)^2$ or $4x^2 - 20x + 25$

In general, $(kx+m)^2$ can be written as:

$(kx)^2 + 2(kx)(m) + m^2$ or $k^2x^2 + 2kmx + m^2$

If a quadratic expression is of the form $ax^2 + bx + c$, then:

- The value of a is k^2.
- The value of b is $2km$.
- The value of c is m^2.

We can use this pattern to help us complete the square and solve equations when the squared term x^2 has a coefficient other than 1—for example, $16x^2 + 40x = 11$.

What constant term c can we add to make the expression on the left of the equal sign a perfect square? And how do we write this expression as a squared factor?

- $a = 16$, which is 4^2, so $k = 4$, and the squared factor could be $(4x + m)^2$.
- 40 is equal to $2(4m)$, so $2(4m) = 40$, or $8m = 40$. This means that $m = 5$.
- If c is m^2, then $c = 5^2$, or $c = 25$.
- So the expression $16x^2 + 40x + 25$ is a perfect square and is equivalent to $(4x + 5)^2$.

Let's solve the equation $16x^2 + 40x = 11$ by completing the square!

$$16x^2 + 40x = 11$$
$$16x^2 + 40x + 25 = 11 + 25$$
$$(4x + 5)^2 = 36$$

$$4x + 5 = 6 \quad \text{or} \quad 4x + 5 = -6$$
$$4x = 1 \quad \text{or} \quad 4x = -11$$
$$x = \frac{1}{4} \quad \text{or} \quad x = -\frac{11}{4}$$

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