Representing Functions at Rational Inputs

Let's find how quantities are growing or decaying over fractional intervals of time.

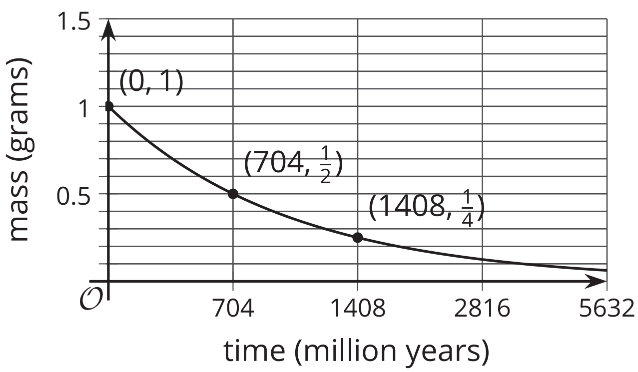
4.1 Radioactive Decay

Many materials such as radioactive elements or large molecules naturally break down over time at a rate that is proportional to how much of the material there is. The amount of material left over time is described mathematically using exponential decay.

To get a sense of how stable the materials are, the length of time it takes for half of the material to remain, or its **half-life**, is given. Here are some half-lives of a few radioactive materials and graphs of how much of the materials remain after there is 1 gram of the material.

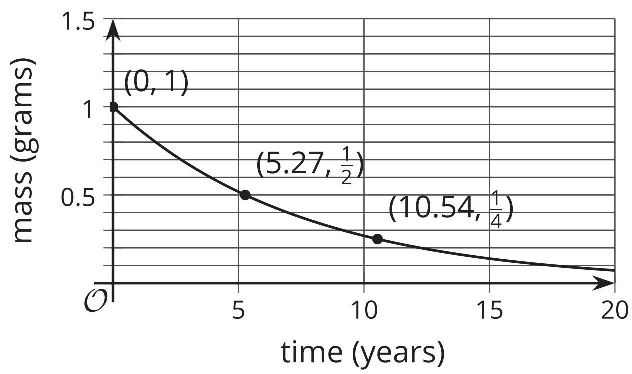
Uranium-235

Half-life: 704 million years



Cobalt-60

Half-life: 5.27 years

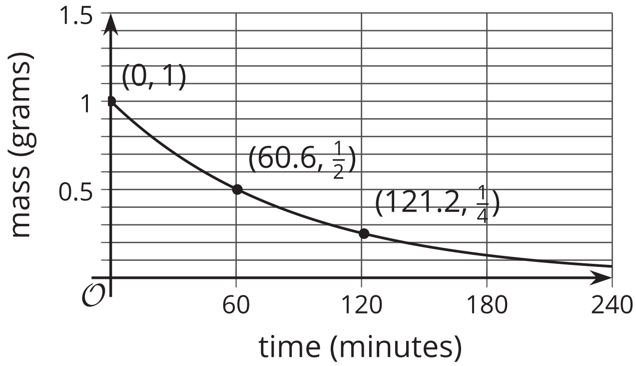


Bismuth-212

Half-life: 60.6 minutes

1. Which of these materials takes the longest to break down?

2. If you had a sample of 8 grams of cobalt-60, how long would it take for it to break down into only 1 gram left? Explain your reasoning.



4.2 Population of Nigeria

In 1990, Nigeria had a population of about 95.3 million. By 2000, there were about 122.4 million people, an increase of about 28.4%. During that decade, the population can be reasonably modeled by an exponential function.

1. Express the population of Nigeria $f(d)$, in millions of people, d decades since 1990.

2. Write an expression to represent the population of Nigeria in 1996.

3. A student said, “The population of Nigeria grew at a rate of 2.84% every year because

$\frac{28.4}{10} = 2.84$."

a. Explain or show why the student's statement is incorrect.

b. Find the correct annual growth rate. Explain or show your reasoning.

4.3 Waiting for Waste

Cesium-137 is a radioactive material found in the waste of nuclear reactors. It has a half-life of about 30 years. Let's suppose that there are 100 grams of cesium-137 as part of some nuclear waste.

1. Each of these expressions describes the amount of cesium-137 in the nuclear waste some number of years after it is produced. For each expression, find the number of years it would take to have that much cesium-137 left for this waste.
   a. $100 \cdot \left(\frac{1}{2}\right)^1$
   b. $100 \cdot \left(\frac{1}{2}\right)^3$
   c. $100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{30}}$
   d. $100 \cdot \left(\frac{1}{2}\right)^t$
2. a. Write a function g to represent the amount of cesium-137 left in the waste, y years after it is produced.

   b. The function f represents the amount of cesium-137 left in the waste after t 30-year periods. Write an equation to represent the function f.

   c. Explain why $g(30) = f(1)$.

Are you ready for more?

What percentage of the initial amount of cesium-137 do you expect to break down in the first 15 years: less than 25%, exactly 25%, or more than 25%? Explain or show your reasoning.

Lesson 4 Summary

Imagine a material has a **half-life** of 3 hours. This means that the amount of material left after 3 hours is half of what there was when it started. For example, if there are 200 milligrams of the material at noon, then at 3 o'clock there will be 100 milligrams left, and at 6 o'clock there will be 50 milligrams left.

If a scientist has 200 mg of the material, then the amount of material, in mg, can be modeled by the function $f(t) = 200 \cdot \left(\frac{1}{2}\right)^t$. In this model, t represents a unit of time. Notice that the 200 represents the initial amount of material the scientist has. The number $\frac{1}{2}$ indicates that for every 1 unit of time, the amount of material is cut in half. Because the half-life is 3 hours, this means that t must measure time in groups of 3 hours.

But what if we wanted to find the amount of material the scientist has each hour after taking it? We know there are 3 equal groups of 1 hour in a 3-hour period. We also know that because the material decays exponentially, it decays by the same factor in each of those intervals. In other words, if b is the decay factor for each hour, then $b \cdot b \cdot b = \frac{1}{2}$, or $b^3 = \frac{1}{2}$. This means that over each hour, the medicine must decay by a factor of $\sqrt[3]{\frac{1}{2}}$, which can also be written as $\left(\frac{1}{2}\right)^{\frac{1}{3}}$. So if h is time in hours since the scientist had 200 mg of the material, we can express the amount of material in mg, g, the scientist has as $g(h) = 200 \cdot \left(\sqrt[3]{\frac{1}{2}}\right)^h$, or $g(h) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{h}{3}}$.