



# Representing Functions at Rational Inputs

Let's find how quantities are growing or decaying over fractional intervals of time.

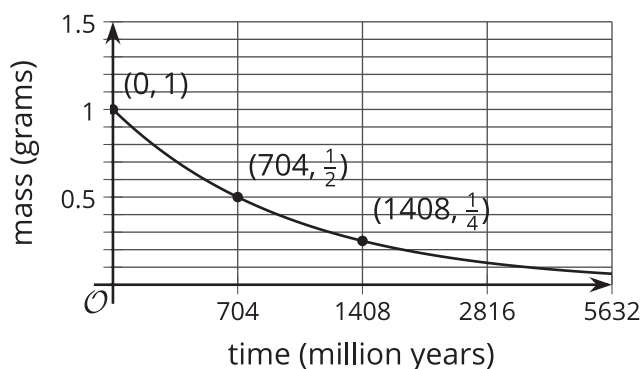
## 4.1 Radioactive Decay

Many materials such as radioactive elements or large molecules naturally break down over time at a rate that is proportional to how much of the material there is. The amount of material left over time is described mathematically using exponential decay.

To get a sense of how stable the materials are, the length of time it takes for half of the material to remain, or its **half-life**, is given. Here are some half-lives of a few radioactive materials and graphs of how much of the materials remain after there is 1 gram of the material.

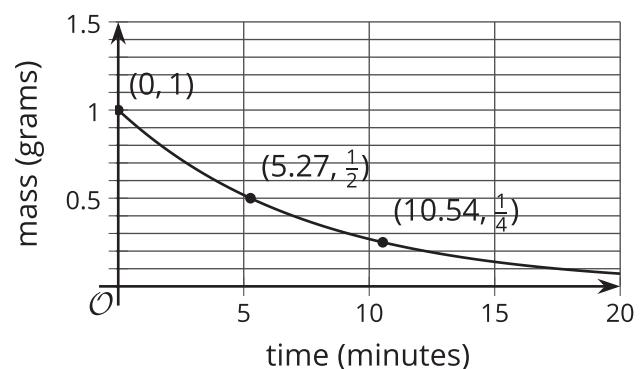
Uranium-235

Half-life: 704 million years



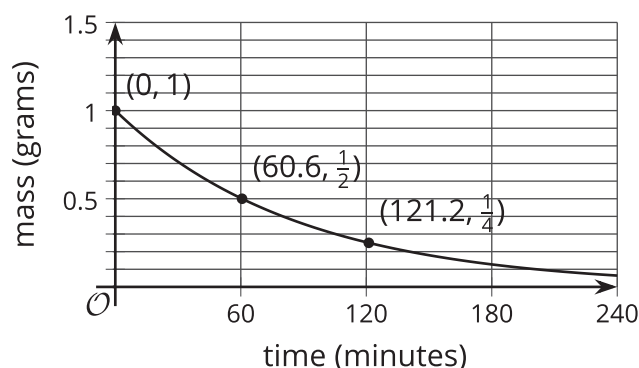
Cobalt-60

Half-life: 5.27 years



Bismuth-212

Half-life: 60.6 minutes



1. Which of these materials takes the longest to break down?
2. If you had a sample of 8 grams of cobalt-60, how long would it take for it to break down into only 1 gram left? Explain your reasoning.

## 4.2 Population of Nigeria



In 1990, Nigeria had a population of about 95.3 million. By 2000, there were about 122.4 million people, an increase of about 28.4%. During that decade, the population can be reasonably modeled by an exponential function.

1. Express the population of Nigeria  $f(d)$ , in millions of people,  $d$  decades since 1990.
2. Write an expression to represent the population of Nigeria in 1996.
3. A student said, “The population of Nigeria grew at a rate of 2.84% every year because  $\frac{28.4}{10} = 2.84$ .”
  - a. Explain or show why the student’s statement is incorrect.
  - b. Find the correct annual growth rate. Explain or show your reasoning.

## 4.3 Waiting for Waste

Cesium-137 is a radioactive material found in the waste of nuclear reactors. It has a half-life of about 30 years. Let’s suppose that there are 100 grams of cesium-137 as part of some nuclear waste.

1. Each of these expressions describes the amount of cesium-137 in the nuclear waste some number of years after it is produced. For each expression, find the number of years it would take to have that much cesium-137 left for this waste.

- a.  $100 \cdot \left(\frac{1}{2}\right)^1$
  - b.  $100 \cdot \left(\frac{1}{2}\right)^3$
  - c.  $100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{30}}$
  - d.  $100 \cdot \left(\frac{1}{2}\right)^t$
2.
    - a. Write a function  $g$  to represent the amount of cesium-137 left in the waste,  $y$  years after it is produced.
    - b. The function  $f$  represents the amount of cesium-137 left in the waste after  $t$  30-year periods. Write an equation to represent the function  $f$ .
    - c. Explain why  $g(30) = f(1)$ .

### Are you ready for more?

What percentage of the initial amount of cesium-137 do you expect to break down in the first 15 years: less than 25%, exactly 25%, or more than 25%? Explain or show your reasoning.

### Lesson 4 Summary

Imagine a material has a **half-life** of 3 hours. This means that the amount of material left after 3 hours is half of what there was when it started. For example, if there are 200 milligrams of the material at noon, then at 3 o'clock there will be 100 milligrams left, and at 6 o'clock there will be 50 milligrams left.

If a scientist has 200 mg of the material, then the amount of material, in mg, can be modeled by the function  $f(t) = 200 \cdot \left(\frac{1}{2}\right)^t$ . In this model,  $t$  represents a unit of time. Notice that the 200 represents the initial amount of material the scientist has. The number  $\frac{1}{2}$  indicates that for every 1 unit of time, the amount of material is cut in half. Because the half-life is 3 hours, this means that  $t$  must measure time in groups of 3 hours.

But what if we wanted to find the amount of material the scientist has each hour after taking it? We know there are 3 equal groups of 1 hour in a 3-hour period. We also know that because the material decays exponentially, it decays by the same factor in each of those intervals. In other words, if  $b$  is the decay factor for each hour, then  $b \cdot b \cdot b = \frac{1}{2}$ , or  $b^3 = \frac{1}{2}$ . This means that over each hour, the medicine must decay by a factor of  $\sqrt[3]{\frac{1}{2}}$ , which can also be written as  $\left(\frac{1}{2}\right)^{\frac{1}{3}}$ . So if  $h$  is time in hours since the scientist had 200 mg of the material, we can express the amount of

material in mg,  $g$ , the scientist has as  $g(h) = 200 \cdot \left(\sqrt[3]{\frac{1}{2}}\right)^h$ , or  $g(h) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{h}{3}}$ .

