Exponential Functions with Base e

Let's look at situations that can be modeled using exponential functions with base e.

13.1 e on a Calculator

e is a mathematical constant whose value is approximately 2.718. When working on problems that involve e, we often rely on calculators to estimate values.

1. Find how to represent e on your calculator. Experiment with it to understand how it works. (For example, see how the value of $2e$ or e^2 can be calculated.)
2. Evaluate each expression. Make sure that your calculator gives the indicated value. If it doesn't, check in with your partner to compare how you entered the expression.
   a. $10 \cdot e^{(1.1)}$ should give approximately 30.04166
   b. $5 \cdot e^{(1.1)(7)}$ should give approximately 11,041.73996
   c. $e^{\frac{9}{23}} + 7$ should give approximately 8.47891

13.2 Same Situation, Different Equations

The population of a colony of insects is 9 thousand when it was first being studied. The two students who are studying the colony of insects choose to model the population in slightly different ways. Here are their two functions used to model the growth of the colony t months after the study began.

$$P(t) = 9 \cdot (1.02)^t \qquad Q(t) = 9 \cdot e^{(0.02t)}$$

1. For each model, use technology to find the population of the colony at different times after the beginning of the study, and complete the table.

t (time in months)	$P(t)$ (population in thousands)	$Q(t)$ (population in thousands)
6		
12		
24		
48		
100		

2. What do you notice about the populations in the two models?

3. Here are pairs of equations representing the populations, in thousands, of four other insect colonies in a research lab. The initial population of each colony is 10 thousand and they are growing exponentially. t is time, in months, since the study began.

Colony 1
$f(t) = 10 \cdot (1.05)^t$
$g(t) = 10 \cdot e^{(0.05t)}$

Colony 2
$k(t) = 10 \cdot (1.03)^t$
$l(t) = 10 \cdot e^{(0.03t)}$

Colony 3
$r(t) = 10 \cdot (1.01)^t$
$q(t) = 10 \cdot e^{(0.01t)}$

Colony 4
$v(t) = 10 \cdot (1.005)^t$
$w(t) = 10 \cdot e^{(0.005t)}$

   a. Graph each pair of functions on the same coordinate plane. Adjust the graphing window to these boundaries to start: $0 < x < 50$, and $0 < y < 80$.

   b. What do you notice about the graph of the equation written using e and the counterpart written without e? Make a couple of observations.

13.3 e in Exponential Models

Exponential models that use e often use the format shown in this example:

$$P(t) = 13\,e^{0.045t}$$

- output of the function: $P(t)$
- value of the function when t is 0: 13
- 0.045 or 4.5% is the continuous growth rate per unit time.
- the constant e (approximately 2.718)
- input to the function (the independent variable): t

Here are some situations in which a percent change is considered to be happening continuously. For each function, complete any missing parts of the function and identify the growth rate as a percentage if it is not given.

1. At time $t = 0$, measured in hours, a scientist puts 50 bacteria into a gel on a dish. The bacteria are growing and the population is expected to show exponential growth.
    - function: $b(t) = 50 \cdot e^{(0.25t)}$
    - continuous growth rate per hour:

2. In 1964, the population of the United States was growing at a rate of 1.4% annually. That year, the population was approximately 192 million. The model predicts the population, in millions, t years after 1964.
    - function: $p(t) =$ ________ $\cdot\ e^{\_\_\_\_ t}$
    - continuous growth rate per year: 1.4%

3. In 1955, the world population was about 2.5 billion and growing. The model predicts the population, in billions, t years after 1955.
    - function: $q(t) =$ ________ $\cdot\ e^{(0.0168t)}$
    - continuous growth rate per year:

13.4 Graphing Exponential Functions with Base e

1. Use graphing technology to graph the function defined by $f(t) = 50 \cdot e^{(0.25t)}$. Adjust the graphing window as needed to answer these questions:
   a. The function f models the population of bacteria t hours after it was initially measured. About how many bacteria are in the dish 10 hours after the scientist put the initial 50 bacteria in the dish?

   b. About how many hours does it take for the number of bacteria in the dish to double? Explain or show your reasoning.

2. Use graphing technology to graph the function defined by $p(t) = 192 \cdot e^{(0.014t)}$. Adjust the graphing window as needed to answer these questions:
   a. The equation models the population, in millions, in the U.S. t years after 1964. What does the model predict for the population of the U.S. in 1974?

   b. In which year does the model predict the population will reach 300 million?

Are you ready for more?

1. In what year does this model predict the population of the United States to be 300 million people? Research what the actual population of the U.S. was in that year. When did the U.S. actually reach 300 million people?

2. Compare the actual U.S. population to the model's predictions. For what years would you say the model could be used to give an estimate of the actual population?

Lesson 13 Summary

Suppose 24 square feet of a pond is covered with algae, and the area is growing at a rate of 8% each day.

The area, in square feet, can be modeled with a function such as $a(d) = 24 \cdot (1 + 0.08)^d$ or $a(d) = 24 \cdot (1.08)^d$, where d is the number of days since the area was 24 square feet. This model assumes that the growth rate of 0.08 happens once each day.

In this lesson, we looked at a different type of exponential function, using the base e. For the algae growth, this might look like $A(d) = 24 \cdot e^{(0.08d)}$. This model is different because the 8% growth per day is not just applied at the end of each day: It is successively divided up and applied at every moment. Because the growth is applied at every moment, or "continuously," the functions a and A are not the same, but the smaller the growth rate the closer they are to each other.

Many functions that express real-life exponential growth or decay are expressed in the form that uses e. For the algae model A, 0.08 per day is called the *continuous growth rate* while $e^{0.08}$ is the growth factor for 1 day. In general, when we express an exponential function in the form $P \cdot e^{rt}$, we are assuming that the growth rate (or decay rate) r is being applied continuously and that e^r is the growth (or decay) factor. When r is small, e^{rt} is close to $(1 + r)^t$.