Sums and Products of Rational and Irrational Numbers

Let's make convincing arguments about why the sums and products of rational numbers and irrational numbers are always certain kinds of numbers.

21.1 Operations on Integers

Here are some examples of integers:

-25 -10 -2 -1 0 5 9 40

1. Experiment with adding any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   a. add up to another integer.
   b. add up to a number that is *not* an integer.
2. Experiment with multiplying any two numbers from the list (or other integers of your choice). Try to find one or more examples of two integers that:
   a. multiply to make another integer.
   b. multiply to make a number that is *not* an integer.

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21.2 Sums and Products of Rational Numbers

1. Here are a few examples of adding two rational numbers. Is each sum a rational number? Be prepared to explain how you know.

    a. $4 + 0.175 = 4.175$

    b. $\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10}$

    c. $-0.75 + \frac{14}{8} = \frac{-6}{8} + \frac{14}{8} = \frac{8}{8} = 1$

    d. a is an integer: $\frac{2}{3} + \frac{a}{15} = \frac{10}{15} + \frac{a}{15} = \frac{10+a}{15}$

2. Here is a way to explain why the sum of two rational numbers is rational:

    Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are fractions. That means that $a, b, c,$ and d are integers, and b and d are not 0.

    a. Find the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Show your reasoning.

    b. In the sum, are the numerator and the denominator integers? How do you know?

    c. Use your responses to explain why the sum of $\frac{a}{b} + \frac{c}{d}$ is a rational number.

3. Use the same reasoning as in the previous question to explain why the product of two rational numbers, $\frac{a}{b} \cdot \frac{c}{d}$, must be rational.

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Are you ready for more?

Consider numbers that are of the form $a + b\sqrt{5}$, where a and b are integers. Let's call such numbers *quintegers*.

Here are some examples of quintegers:

- $3 + 4\sqrt{5}$ $(a = 3, b = 4)$
- $7 - 2\sqrt{5}$ $(a = 7, b = \text{-}2)$
- $\text{-}5 + \sqrt{5}$ $(a = \text{-}5, b = 1)$
- 3 $(a = 3, b = 0)$.

1. When we add two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose sum is not a quinteger.

2. When we multiply two quintegers, will we always get another quinteger? Either prove this or find two quintegers whose product is not a quinteger.

21.3 Sums and Products of Rational and Irrational Numbers

1. Here is a way to explain why $\sqrt{2} + \frac{1}{9}$ is irrational.
   - Let s be the sum of $\sqrt{2}$ and $\frac{1}{9}$, or $s = \sqrt{2} + \frac{1}{9}$.
   - Suppose s is rational.

   a. Is $s + -\frac{1}{9}$ rational or irrational? Explain how you know.

   b. Evaluate $s + -\frac{1}{9}$. Is the sum rational or irrational?

   c. Use your responses to explain why s cannot be a rational number, and therefore $\sqrt{2} + \frac{1}{9}$ cannot be rational.

2. Use a similar reasoning as in the earlier question to explain why $\sqrt{2} \cdot \frac{1}{9}$ is irrational. Here are some assumptions to get you started.
   - Let p be the product of $\sqrt{2}$ and $\frac{1}{9}$, or $p = \sqrt{2} \cdot \frac{1}{9}$.
   - Suppose p is rational.

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21.4 Equations with Different Kinds of Solutions

1. Consider the equation $4x^2 + bx + 9 = 0$. Find a value of b so that the equation has:

   a. 2 rational solutions

   b. 2 irrational solutions

   c. 1 solution

   d. no solutions

2. Describe all the values of b that produce 2 solutions, 1 solution, and no solutions.

3. Write a new quadratic equation with each type of solution. Be prepared to explain how you know that your equation has the specified type and number of solutions.

   a. no solutions

   b. 2 irrational solutions

   c. 2 rational solutions

   d. 1 solution

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Lesson 21 Summary

We know that quadratic equations can have rational solutions or irrational solutions. For example, the solutions to $(x+3)(x-1)=0$ are -3 and 1, which are rational. The solutions to $x^2-8=0$ are $\pm\sqrt{8}$, which are irrational.

Sometimes solutions to equations combine two numbers by addition or multiplication—for example, $\pm 4\sqrt{3}$ and $1+\sqrt{12}$. What kind of numbers are these expressions?

When we add or multiply two rational numbers, is the result rational or irrational?

- The sum of two rational numbers is rational. Here is one way to explain why it is true:
    - Any two rational numbers can be written $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c,$ and d are integers, and b and d are not zero.
    - The sum of $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad+bc}{bd}$. The denominator is not zero because neither b nor d is zero.
    - Multiplying or adding two integers always gives an integer, so we know that ad, bc, bd and $ad+bc$ are all integers.
    - If the numerator and denominator of $\frac{ad+bc}{bd}$ are integers, then the number is a fraction, which is rational.
- The product of two rational numbers is rational. We can show why in a similar way:
    - For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $a, b, c,$ and d are integers, and b and d are not zero, the product is $\frac{ac}{bd}$.
    - Multiplying two integers always results in an integer, so both ac and bd are integers. Therefore, $\frac{ac}{bd}$ is a rational number.

What about two irrational numbers?

- The sum of two irrational numbers could be either rational or irrational. We can show this through examples:
    - $\sqrt{3}$ and -$\sqrt{3}$ are both irrational, but their sum is 0, which is rational.
    - $\sqrt{3}$ and $\sqrt{5}$ are both irrational, and their sum is irrational.
- The product of two irrational numbers could be either rational or irrational. We can show this through examples:
    - $\sqrt{2}$ and $\sqrt{8}$ are both irrational, but their product is $\sqrt{16}$, or 4, which is rational.
    - $\sqrt{2}$ and $\sqrt{7}$ are both irrational, and their product is $\sqrt{14}$, which is not a perfect square and is therefore irrational.

What about a rational number and an irrational number?

- The sum of a rational number and an irrational number is irrational. To explain why requires a slightly different argument:
    - Let R be a rational number and I an irrational number. We want to show that $R + I$ is irrational.
    - Suppose s represents the sum of R and I ($s = R + I$), and suppose s is rational.
    - If s is rational, then $s + \text{-}R$ would also be rational, because the sum of two rational numbers is rational.
    - $s + \text{-}R$ is not rational, however, because $(R + I) + \text{-}R = I$.
    - $s + \text{-}R$ cannot be both rational and irrational, which means that our original assumption that s is rational was incorrect. s, which is the sum of a rational number and an irrational number, must be irrational.
- The product of a nonzero rational number and an irrational number is irrational. We can show why this is true in a similar way:
    - Let R be rational and I irrational. We want to show that $R \boldsymbol{\cdot} I$ is irrational.
    - Suppose p is the product of R and I ($p = R \boldsymbol{\cdot} I$), and suppose p is rational.
    - If p is rational, then $p \boldsymbol{\cdot} \frac{1}{R}$ would also be rational because the product of two rational numbers is rational.
    - $p \boldsymbol{\cdot} \frac{1}{R}$ is not rational, however, because $R \boldsymbol{\cdot} I \boldsymbol{\cdot} \frac{1}{R} = I$.
    - $p \boldsymbol{\cdot} \frac{1}{R}$ cannot be both rational and irrational, which means our original assumption that p is rational was false. p, which is the product of a rational number and an irrational number, must be irrational.