Unit 1, Lesson 13

Proofs about Parallelograms

Goals

- Justify (orally) and prove (in writing) that the diagonals of a parallelogram bisect each other.
- Prove (orally and in writing) that if the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Learning Targets

- I can prove theorems about the diagonals of a parallelogram.

Lesson Narrative

In this lesson, students prove two statements about the diagonals of parallelograms.

- The diagonals of a parallelogram bisect each other.
- If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Students learn a new strategy for looking for structure (MP7) by working backward from the statement they are trying to prove to the given statements. Students also encounter a situation where they could use overlapping triangles, which can be challenging. Students learn techniques for redrawing or marking diagrams to help them see additional triangles to use in congruence proofs. As students prove theorems about parallelograms, they are explicitly practicing proof techniques and looking for structure.

One of the activities in this lesson works best when each student has access to internet-enabled devices because students will benefit from seeing the relationship in a dynamic way.

Standards

Building On	**HSG-CO.B.8, HSG-CO.C.9**
Addressing	**HSG-CO.C.11**
Building Toward	**HSG-CO.C.11**

Instructional Routines

- Draw It
- MLR1: Stronger and Clearer Each Time
- MLR2: Collect and Display
- Notice and Wonder
- Take Turns

Required Materials

Materials to Gather

- Internet-enabled device: Activity 1
- 1-inch strips cut from card stock with evenly spaced holes: Activity 2
- Geometry toolkits (HS): Activity 2
- Metal paper fasteners: Activity 2
- Rulers: Activity 2

Materials to Copy

- Brace Yourself! Short Strips (1 copy for every 1 students): Activity 2
- Brace Yourself! Long Strips (1 copy for every 1 students): Activity 2

Required Preparation

Activity 1:

Acquire internet-enabled devices that can run the applet in “Notice and Wonder: Diagonals,” one for every 2–3 students. If technology is not available, there is a paper and pencil alternative, but consider displaying the applet for all to see.

Activity 2:

Students will need the 1-inch strips with evenly spaced holes and fasteners prepared in an earlier lesson. Make additional copies from the blackline master, as needed.

Student Facing Learning Goals

Let’s prove theorems about parallelograms.

13.1 Notice and Wonder: Diagonals

Warm-up

5 min

Activity Narrative

There is a digital version of this activity.

The purpose of this *Warm-up* is to elicit conjectures about rectangles and parallelograms, which will be useful when students write proofs of these conjectures in a subsequent activity. While students may notice and wonder many things about these images, the relationships between the diagonals of a parallelogram and the diagonals of a rectangle is the important discussion point.

When students articulate what they notice and wonder, they have an opportunity to attend to precision in the language they use to describe what they see (MP6). They might first propose less formal or imprecise language, then restate their observation with more precise language in order to communicate more clearly.

This activity is designed to be done digitally because students will benefit from seeing the relationship in a dynamic way. If students don't have individual access, projecting and interacting with the applet of the parallelogram and rectangle would be helpful during the *Launch* as students notice and wonder.

Standards

Building Toward **HSG-CO.C.11**

Instructional Routines

- Notice and Wonder

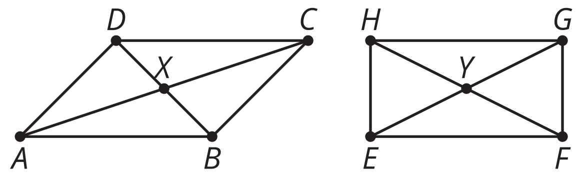
Launch

Display the image for all to see. Ask students to think of at least one thing they notice and at least one thing they wonder. Give students 1 minute of quiet think time and then 1 minute to discuss with their partner the things they notice.

Consider displaying the applet for all to see by navigating to this URL: ggbm.at/cf56pbrv.

Student Task Statement

Here is parallelogram $ABCD$ and rectangle $EFGH$. What do you notice? What do you wonder?



Student Response

Things students may notice:

- The diagonals of the parallelogram are not the same length.
- The diagonals of the rectangle bisect each other.
- The four smaller triangles are not congruent in the rectangle or the parallelogram.
- Pairs of the smaller triangles are congruent in the rectangle and the parallelogram.

Things students may wonder:

- Are the diagonals of the rectangle the same length?
- Do the diagonals of the parallelogram bisect each other?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After all responses have been recorded without commentary or editing, ask students, “Is there anything on this list that you are wondering about now?” Encourage students to respectfully disagree, ask for clarification, or point out contradicting information. If conjectures about diagonals do not come up during the conversation, ask students to discuss this idea.

15 min

13.2 The Diagonals of a Parallelogram

Activity Narrative

This activity invites students to convince themselves, then a partner, and then a skeptic that the diagonals of a parallelogram bisect each other. Students can use transformations or congruent triangles to convince a skeptic that the diagonals of a parallelogram bisect each other.

Stating the goal of the proof in different ways may help students see a different path to the proof. For example, the proof can be restated as “Show that the midpoint of AC and the midpoint of BD are both the point of intersection.” This might suggest a transformation approach based on rotating 180° using the midpoint of AC as the center.

Monitor for different approaches to the proof.

Making dynamic geometry software available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Building On **HSG-CO.B.8, HSG-CO.C.9**

Addressing **HSG-CO.C.11**

Instructional Routines

- Draw It
- MLR1: Stronger and Clearer Each Time

Launch

Arrange students in groups of 2. Provide access to geometry toolkits, rulers, 1-inch strips, and fasteners.

Before they begin to confirm it, briefly discuss the conjecture with students by asking how we could rewrite the conjecture "The diagonals of a parallelogram bisect each other." (The intersection of the diagonals is the midpoint of each diagonal. In parallelogram $ABCD$, where the diagonals intersect at point X, we have that $AX = CX$ and $BX = DX$.)

Access for English Language Learners

MLR1 Stronger and Clearer Each Time. Before the whole-class discussion, give students time to meet with 2–3 partners to share and get feedback on their first draft proof. Invite listeners to ask questions and give feedback that will help their partner clarify and strengthen their ideas and writing. Give students 3–5 minutes to revise their first draft based on the feedback they receive.
Advances: Writing, Speaking, Listening

Student Task Statement

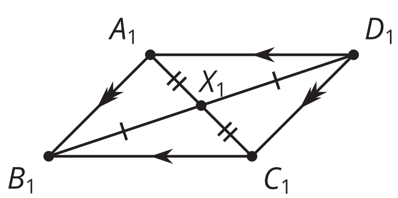
Conjecture: The diagonals of a parallelogram bisect each other.

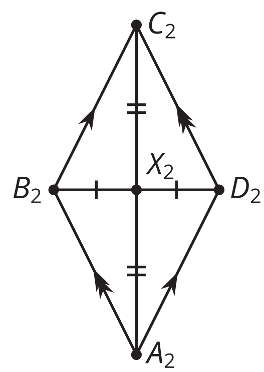
1. Use the tools available to convince yourself the conjecture is true.
2. Convince your partner that the conjecture is true for any parallelogram.
3. What information is needed to prove that the diagonals of a parallelogram bisect each other?
4. Prove that segment AC bisects segment BD, and that segment BD bisects segment AC.

Student Response

Sample responses:

1.





2. We could prove that triangles AXD and CXB are congruent.
3. We are trying to prove that the sides are congruent, so we will probably need to use the Angle-Side-Angle Triangle

Congruence Theorem. We need to prove that at least two pairs of corresponding angles are congruent.

4. Because $ABCD$ is a parallelogram, I know that segments AD and CB are congruent (opposite sides of a parallelogram are congruent) and lines AD and CB are parallel. BD is a transversal of parallel lines AD and CB, so angles ADX and CBX are alternate interior angles, and are therefore congruent. By the same reasoning, angles BCX and DAX are congruent. Therefore, triangles AXD and CXB are congruent by the Angle-Side-Angle Triangle Congruence Theorem. Because corresponding parts of congruent triangles are congruent, segment BX is congruent to segment DX, so AC bisects BD. By the same reasoning, segment AX is congruent to segment CX, so BD bisects AC.

Building on Student Thinking

If students are stuck, offer these questions:

- To what triangle is triangle AXB congruent?
- What do you know about parallelograms? (Encourage students to look at their reference chart.)

Activity Synthesis

Display the proofs of two different groups for all to see.

Ask the class what is the same and what is different. (They both start with the same given information and reach the same conclusion. Their diagrams look different.)

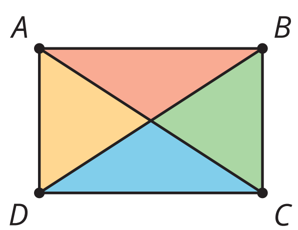
13.3 Work Backward to Prove

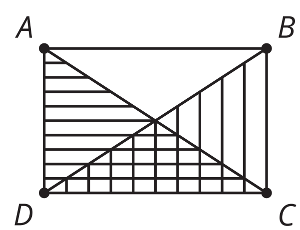
10 min

Activity Narrative

Working backward from what we are trying to show in order to see how that relates to the given information is a useful skill in geometry. This activity makes that strategy explicit by having partners take turns saying, "I would know that were true if . . . ." This is one of the first cases where students might find and use overlapping triangles in a proof. Students can use triangles ADC and BCD, or they can study the angles in triangles CXD and DXA and prove that angles ADX and CDX must be complementary (where X is the intersection of AC and DB). This method will require some algebraic manipulation of expressions for the angles.

Monitor for students who are focused on the four non-overlapping triangles formed by the diagonals, and for students who are focused on the two larger overlapping triangles formed by the diagonals.





This is the first time Math Language Routine 2: *Collect and Display* is suggested in this course. In this routine, the teacher circulates and listens to student talk while jotting down words, phrases, drawings, or writing that students use. The language collected is displayed visually for the whole class to use throughout the lesson and unit. The purpose of this routine is to capture a variety of students' words and phrases—including, especially, everyday or social language and non-English—in a display that students can refer to, build on, or make connections with during future discussions, and to increase students' awareness of language used in mathematics conversations.

Access for English Language Learners

This activity uses the *Collect and Display* math language routine to advance students in developing their mathematical language.

Standards

Building On **HSG-CO.B.8, HSG-CO.C.9**

Addressing **HSG-CO.C.11**

Instructional Routines

- MLR2: Collect and Display
- Take Turns

Launch

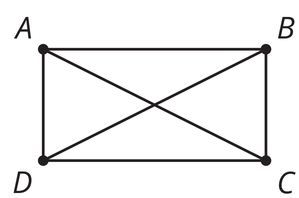
Arrange students in groups of 2. Tell students to make sure to write down each statement since they will be using these statements in the *Cool-down*.

Use *Collect and Display* to create a shared reference using students' developing mathematical language. Collect the language students use to complete the statement. Display words and phrases, such as "right angle," "triangle ACD" and "corresponding parts are congruent."

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Provide prompts or checklists that focus on increasing the length of on-task orientation in the face of distractions. For example, remind students to use the reference chart, to try looking for other triangles, and to redraw the triangles in the same orientation.
Supports accessibility for: Attention, Social-Emotional Functioning

Student Task Statement



Given: $ABCD$ is a parallelogram with AB parallel to CD and AD parallel to BC. Diagonal AC is congruent to diagonal BD.

Prove: $ABCD$ is a rectangle (angles A, B, C, and D are right angles).

With your partner, you will work backward from the statement to the proof until you feel confident that you can prove that $ABCD$ is a rectangle using only the given information.

Start with this sentence: I would know $ABCD$ is a rectangle if I knew ________________.
Then take turns saying this sentence: I would know [what my partner just said in the blank] if I knew ________________.

Write down what each of you say. If you get to a statement and get stuck, go back to an earlier statement and try to take a different path.

Student Response

Sample response: I would know $ABCD$ is a rectangle if I knew the angles were right angles.

I would know angle ABC and DCB were right angles if I knew they were congruent. (And then I would know the other angles were right angles too because opposite angles of a parallelogram are congruent.)

I would know angle ABC was congruent to angle DCB if I knew they were part of two congruent triangles.

I would know angles ABC and DCB were part of two congruent triangles if I knew triangles ABC and DCB were congruent.

I would know triangles ABC and DCB were congruent if I could use the Side-Side-Side Triangle Congruence Theorem.

I would know I could use the Side-Side-Side Triangle Congruence Theorem if I knew that side AB was congruent to side DC, diagonal AC was congruent to diagonal DB, and side BC was congruent to side CB.

I know side AB is congruent to side DC because it is given that $ABCD$ is a parallelogram.

I know diagonal AC is congruent to diagonal DB because it is given.

I know side BC is congruent to side CB because it's the same segment.

Building on Student Thinking

If students get stuck using the small triangles, suggest they try looking for other triangles.

If students get mixed up using the overlapping triangles, suggest they redraw the triangles outside of the rectangle.

Encourage students who are struggling to come up with reasons to use their reference chart to find statements that justify what they are trying to show.

Are You Ready for More?

Two intersecting segments always make a quadrilateral if the endpoints are connected. What has to be true about the intersecting segments in order to make a(n):

1. rectangle
2. rhombus
3. square
4. kite
5. isosceles trapezoid

Extension Student Response

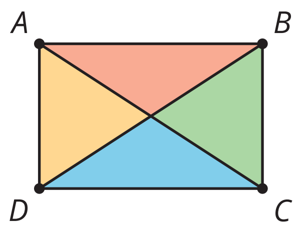
Sample responses:

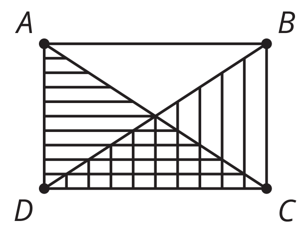
1. The segments have the same length and bisect each other.
2. The segments are perpendicular and bisect each other.
3. The segments have the same length, are perpendicular, and bisect each other.
4. The segments are perpendicular, and one segment bisects the other segment.

5. The segments have the same length.

Activity Synthesis

Direct students' attention to the reference created using *Collect and Display*. Ask students to share which triangles they used. Invite students to borrow language from the display as needed, and update the reference to include additional phrases as they respond.





Encourage students who share to use different colors or redraw the triangles on a display for the whole class to see. Discuss which triangles were easier to see and which turned out to be most useful in the proof. Remind students that the easiest triangles to use might be overlapping or hidden in some way.

Lesson Synthesis

Remind students of the term "converse." Ask students what the converse of "If a quadrilateral is a parallelogram, then its diagonals bisect each other" would be. (If a quadrilateral's diagonals bisect each other, then it is a parallelogram.) We have proven both the statement and its converse. Introduce the phrase "if and only if," and explain that we use it when the statement *and* its converse are true. Display these statements for all to see:

1. A quadrilateral is a parallelogram if its diagonals bisect each other.
2. A quadrilateral is a parallelogram *only if* its diagonals bisect each other.

Ask multiple students to put the second statement in their own words. (There can't be any parallelograms with diagonals that don't bisect each other; all quadrilaterals have diagonals that bisect each other; if it's a parallelogram, then the diagonals bisect each other).

Use some more everyday examples, and ask students which are examples of if-and-only-if statements (the statement and its converse are true) and which are not. For example:

1. A person is a parent if and only if they are a mother. (This is not an if-and-only-if statement. A father can be a parent too. It's true that if they're a mother, then they're a parent, but it's not true that if they're a parent, then they must be a mother.)
2. A person is vegan if and only if they don't eat animal products. (This is as close to true as things get in the real world, since it's the definition of being vegan.)

Invite students to come up with their own examples and nonexamples. Ask students why it is useful to know a quadrilateral is a parallelogram if and only if its diagonals bisect each other. (Now we can use either part as the given information, so it's like adding two theorems to the reference chart.)

13.4 Backward to Forward

Cool-down

10 min

Access for English Language Learners

This activity uses the *Collect and Display* math language routine to advance students in developing their mathematical language.

Standards

Building On HSG-CO.B.8, HSG-CO.C.9

Addressing HSG-CO.C.11

Launch

Use *Collect and Display* to direct attention to words collected and displayed from an earlier activity. Invite students to borrow language from the display as needed.

Student Task Statement

Use your notes from the "Work Backwards to Prove" activity to write a proof that if the diagonals of a parallelogram are congruent, the parallelogram must be a rectangle.

Given: $ABCD$ is a parallelogram with AB parallel to CD and AD parallel to BC. Diagonal AC is congruent to diagonal BD.

Prove: $ABCD$ is a rectangle (angles A, B, C, and D are right angles).

Student Response

Sample response: I know BC is congruent to CB because it's the same segment. I know AC is congruent to DB because it's given. I know AB is congruent to DC because $ABCD$ is a parallelogram (given) and opposite sides of a parallelogram are congruent. Because AB is congruent to DC, AC is congruent to DB, and BC is congruent to CB, by the Side-Side-Side Triangle Congruence Theorem, triangles ABC and DCB are congruent. Angle ABC is congruent to angle DCB because they are corresponding parts of two congruent triangles. Angles ABC and DCB are right angles because they're congruent and supplementary (because they are adjacent angles in a parallelogram). Congruent supplementary angles must be right angles. Opposite angles in a parallelogram are congruent, so if angles ABC and DCB are right angles, then angles DAB and CDA must be too. I know $ABCD$ is a rectangle because angles ABC, DCB, DAB, and CDA are all right angles, and a quadrilateral with four right angles is a rectangle.

Responding to Student Thinking

More Chances

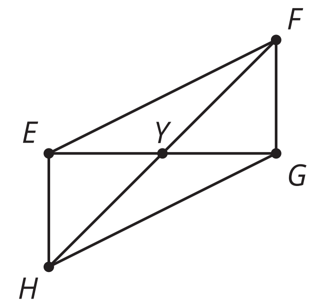
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Lesson 13 Summary

A quadrilateral is a parallelogram if and only if its diagonals bisect each other. The “if and only if” language means that both the statement and its *converse* are true. So we need to prove:

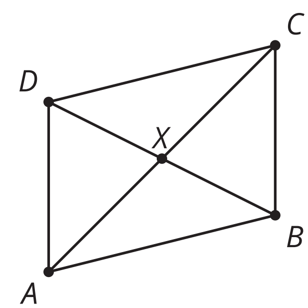
1. If a quadrilateral has diagonals that bisect each other, then it is a parallelogram.
2. If a quadrilateral is a parallelogram, then its diagonals bisect each other.

To prove part 1, make the statement specific: If quadrilateral $EFGH$ has diagonals EG and FH that intersect at Y such that EY is congruent to YG and FY is congruent to YH, then side EF is parallel to side GH, and side EH is parallel to side FG.



We could prove triangles EYH and GYF are congruent by the Side-Angle-Side Triangle Congruence Theorem. That means that corresponding angles in the triangles are congruent, so angle YEH is congruent to YGF. This means that alternate interior angles formed by lines EH and FG are congruent, so lines EH and FG are parallel. We could also make an argument that shows triangles EYF and GYH are congruent. Then, angles FEY and HGY are congruent, which means that lines EF and GH must be parallel.

To prove part 2, make the statement specific: If parallelogram $ABCD$ has side AB parallel to side CD and side AD parallel to side BC, and diagonals AC and BD that intersect at X, then we are trying to prove that X is the midpoint of AC and of BD.



We could use a transformation proof. Rotate parallelogram $ABCD$ by 180° using the midpoint of diagonal AC as the center of the rotation. Then show that the midpoint of diagonal AC is also the midpoint of diagonal BD. That point must be X since it is the only point on both line AC and line BD. So X must be the midpoints of both diagonals, meaning the diagonals bisect each other.

We have proved that any quadrilateral with diagonals that bisect each other is a parallelogram, and that any parallelogram has diagonals that bisect each other. Therefore, a quadrilateral is a parallelogram *if and only if* its diagonals bisect each other.

Lesson 13 Practice Problems

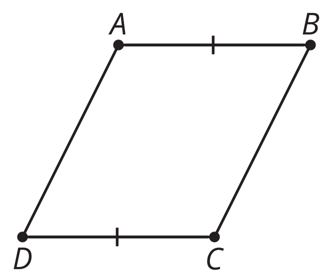
1 Student Task Statement

Conjecture: A quadrilateral with one pair of sides both congruent and parallel is a parallelogram.

a. Draw a diagram of the situation.
b. Mark the given information.
c. Restate the conjecture as a specific statement using the diagram.

Solution

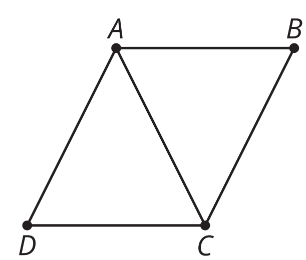
Sample response:



In quadrilateral $ABCD$, AB is congruent to CD, and AB is parallel to CD. Show that $ABCD$ is a parallelogram.

2 Student Task Statement

In quadrilateral $ABCD$, AD is congruent to BC, and AD is parallel to BC. Show that $ABCD$ is a parallelogram.



Solution

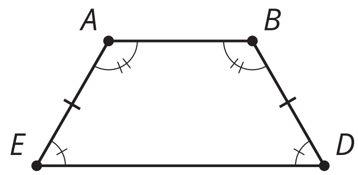
Since AD is parallel to BC, alternate interior angles DAC and BCA are congruent. AC is congruent to AC since segments are congruent to themselves. Along with the given information that AD is congruent to BC, triangle ADC is congruent to CBA by SAS Triangle Congruence. Since the triangles are congruent, all pairs of corresponding angles are congruent, so angle DCA is congruent to BAC. Since those alternate interior angles are congruent, AB must be parallel to CD. Since we define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, $ABCD$ is a parallelogram.

3 from Unit 1, Lesson 12

Student Task Statement

$ABDE$ is an isosceles trapezoid. Name one pair of congruent triangles that could be used to show that the diagonals of an isosceles trapezoid are congruent.

$\overline{AE} \cong \overline{BD}, \angle A \cong \angle B, \angle E \cong \angle D$



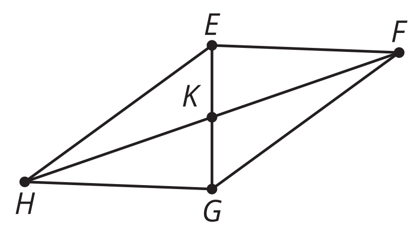
Solution

Sample responses: triangles ABE and BAD, triangles AED and BDE

4 from Unit 1, Lesson 12

Student Task Statement

Select the conjecture with the rephrased statement of proof to show that the diagonals of a parallelogram bisect each other.



A. In parallelogram $EFGH$, show triangle HEF is congruent to triangle FGH.

B. In parallelogram $EFGH$, show triangle EKH is congruent to triangle GKF.

C. In parallelogram $EFGH$, show EK is congruent to KG and FK is congruent to KH.

D. In quadrilateral $EFGH$ with GH congruent to FE and EH congruent to FG, show $EFGH$ is a parallelogram.

Solution

C

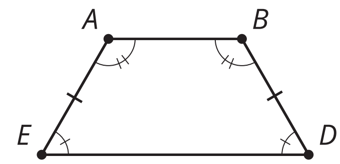
5 from Unit 1, Lesson 12

Student Task Statement

Priya is convinced the diagonals of the isosceles trapezoid are congruent. She knows that if she can prove triangles that include the diagonals congruent, then she will show that diagonals are also congruent. Help her complete the proof.

$ABDE$ is an isosceles trapezoid.

$\overline{AE} \cong \overline{BD}, \angle A \cong \angle B, \angle E \cong \angle D$



Draw auxiliary lines that are diagonals ____1____ and ____2____. AB is congruent to ____3____ because they are the same segment. We know angle B and ____4____ are congruent. We know AE is congruent to ____5____. Therefore, triangle ABE and ____6____ are congruent because of ____7____. Finally, diagonal BE is congruent to ____8____ because ____9____.

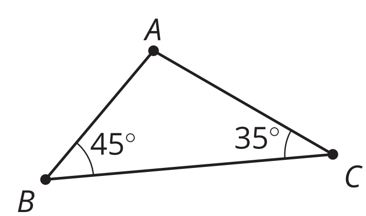
Solution

1. AD or BE
2. AD or BE (whichever was not used in question 1)
3. BA
4. angle A
5. BD
6. triangle BAD
7. the Side-Angle-Side Triangle Congruence Theorem
8. AD
9. corresponding parts of congruent figures are congruent

6 from Unit 1, Lesson 4

Student Task Statement

List the side lengths in order from least to greatest.



Solution

$AB < AC < BC$