



From Parallelograms to Triangles

Goals

- Describe (orally and in writing) ways in which two identical triangles can be composed, i.e., into a parallelogram or into a rectangle.
- Show how any parallelogram can be decomposed into two identical triangles by drawing a diagonal, and generalize (in writing) that this property applies to all parallelograms, but not all quadrilaterals.

Learning Targets

- I can explain the special relationship between a pair of identical triangles and a parallelogram.

Lesson Narrative

This lesson prepares students to apply what they know about the area of parallelograms to reason about the area of triangles.

Highlighting the relationship between triangles and parallelograms is a key goal of this lesson. The activities make use of both the idea of *decomposition* (of a quadrilateral into triangles) and *composition* (of two triangles into a quadrilateral). The two-way study is designed to help students view and reason about the area of a triangle differently and to look for structure (MP7). Students see that a parallelogram can always be decomposed into two identical triangles, and that any two identical triangles can always be composed into a parallelogram.

Because a lot happens in this lesson and timing might be tight, it is important to both prepare all the materials and consider grouping arrangements in advance.

Standards

Addressing 6.G.A.1
Building Toward 6.G.A.1

Instructional Routines

- MLR2: Collect and Display

Required Materials

Materials to Gather

- Geometry toolkits: Activity 1, Activity 2
- Rulers: Activity 2

Materials to Copy

- A Tale of Two Triangles (Part 2) Cutouts (1 copy for every 3 students): Activity 3

Required Preparation

Activity 3:

Print pairs of triangles from the blackline master. If students are cutting out the triangles, use the first page only. If the triangles are to be pre-cut by the teacher, print the second and third pages. Prepare enough sets so that each group of 3–4 students has a complete set of 6 pairs of triangles labeled P, Q, R, S, T, and U (2 copies of each).

For the digital version of the activity, acquire devices that can run the applet.



Student Facing Learning Goals

 Let's compare parallelograms and triangles.

6.1 Same Parallelograms, Different Bases

Warm-up

 5 min

Activity Narrative

This *Warm-up* reinforces students' understanding of bases and heights in a parallelogram. In previous lessons, students calculated areas of parallelograms using bases and heights. They have also determined possible bases and heights of a parallelogram given a whole-number area. They saw, for instance, that finding possible bases and corresponding heights of a parallelogram with an area of 20 square units means finding two numbers with a product of 20. Students extend that work here by working with decimal side lengths and area.

As students work, notice students who understand that the two identical parallelograms have equal area and who use that understanding to find the unknown base. Ask them to share later.

Standards

Addressing 6.G.A.1

Launch

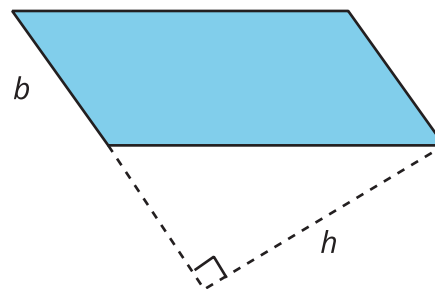
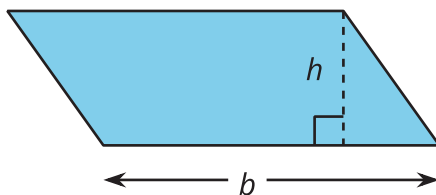
Give students 2 minutes of quiet work time and access to their geometry toolkits.

Students should be adequately familiar with bases and heights to begin the warm-up. If needed, however, briefly review the relationship between a pair of base and height in a parallelogram, using questions such as:

- “Can we use any side of a parallelogram as a base?” (Yes.)
- “Is the height always the length of one of the sides of the parallelogram?” (No.)
- “Once we have identified a base, how do we identify a height?” (It can be any segment that is perpendicular to the base and goes from the base to the opposite side.)
- “Can a height segment be drawn outside of a parallelogram?” (Yes.)

Student Task Statement

Here are two copies of a parallelogram. Each copy has one side labeled as the base b and a segment drawn for its corresponding height and labeled h .



1. The base of the parallelogram on the left is 2.4 centimeters; its corresponding height is 1 centimeter. Find its area in square centimeters.
2. The height of the parallelogram on the right is 2 centimeters. How long is the base of that parallelogram? Explain your reasoning.

Student Response

1. 2.4 square centimeters. $(2.4) \cdot 1 = 2.4$
2. 1.2 centimeters. Sample reasoning: The area of the second parallelogram is also 2.4 square centimeters. Since the base and height must multiply to the same area of 2.4, the base must be 1.2 centimeters because $(1.2) \cdot 2 = 2.4$.

Building on Student Thinking

Some students may not know how to begin answering the questions because measurements are not shown on the diagrams. Ask students to label the parallelograms based on the information in the *Task Statement*.

Students may say that there is not enough information to answer the second question because only one piece of information is known (the height). Ask them what additional information might be needed. Prompt them to revisit the task statement and see what it says about the two parallelograms. Ask what they know about the areas of two figures that are identical.

Students may know what to do to find the unknown base in the second question but be unsure how to divide a number containing a decimal. Ask them to explain how they would reason about it if the area were a whole number. If they understand that they need to divide the area by 2 (because the height is 2 cm and the area is 2.4 sq cm), encourage them to reason in terms of multiplication, for instance by asking, “2 times what number is 2.4?” Or, urge them to consider dividing using fractions, for instance, by seeing 2.4 as $2\frac{4}{10}$ or $\frac{24}{10}$. Ask, “what is 24 tenths divided by 2?”

Activity Synthesis

Select 1–2 previously identified students to share their responses. If not already explained by students, emphasize that we know the parallelograms have the same area because they are identical, which means that when one is placed on top of the other they would match up exactly.

Before moving on, ask students: “How can we verify that the height we found is correct, or that the two pairs of bases and heights produce the same area?” (We can multiply the values of each pair and see if they both produce 2.4.)

6.2

A Tale of Two Triangles (Part 1)

🕒 15 min

Activity Narrative

In this activity, students are given various quadrilaterals and asked to decompose each into two identical triangles by drawing a line segment. They observe the kinds of quadrilaterals for which this is possible. To check whether the two triangles in a quadrilateral are identical, students trace one triangle on tracing paper and then rotate it to match the other triangle. The process prepares students to see any triangle as occupying half of a parallelogram, and consequently, as having one half of its area.

To generalize about quadrilaterals that can be decomposed into identical triangles, students need to analyze the



features of the given shapes and look for structure (MP7).

There are a number of geometric observations in this unit that must be taken for granted at this point in students' study of mathematics. This is one of those instances. Students have seen only examples of a parallelogram being decomposable into two copies of the same triangle, or have verified this conjecture only through physical experimentation, but for the time being it can be considered a fact. Starting in grade 8, they will begin to prove some of the observations they have previously taken to be true.

Standards

Building Toward 6.G.A.1

Instructional Routines

- MLR2: Collect and Display

Launch

Arrange students in groups of 3–4. Give students access to geometry toolkits and 2 minutes of quiet think time for the first two questions. Then, ask them to share their drawings with their group and discuss how they drew their lines. If group members disagree on whether a quadrilateral can be decomposed into two identical triangles, they should note the disagreement, but it is not necessary to come to an agreement. They will soon have a chance to verify their responses.

Next, ask students to use tracing paper to check that the pairs of triangles that they believe to be identical are indeed so. (If placed on top of one another, two triangles that are identical will match up exactly.) Tell students to divide the checking work among the members of their group to optimize time.

Though students have worked with tracing paper earlier in the unit, some may not recall how to use it to check the congruence of two shapes, so some explicit guidance might be needed. Encourage students to work carefully and precisely. A straightedge can be used in tracing but is not essential and may get in the way.

Once students finish checking the triangles in their list and verify that the triangles are identical (or correct their initial response), ask them to answer the last question.

Access for English Language Learners

MLR2 Collect and Display. Collect the language students use to characterize the quadrilaterals they decomposed and the resulting triangles. Display words or phrases, such as: “parallelograms,” “two pairs of parallel sides,” “two pairs of equal sides,” and “the triangles match up exactly.” During the synthesis, invite students to suggest ways to update the display: “What are some other words or phrases we should include?” Invite students to borrow language from the display as needed.

Advances: Conversing, Representing

Access for Students with Disabilities

Action and Expression: Provide Access for Physical Action. Give students who need support with fine-motor skills the option of representing the situation in the activity kinesthetically on a larger scale. For example, enlarge the figures using a copier or make new copies on plain or graph-ruled chart paper. Ask students to draw or show how they would decompose the quadrilaterals into two identical triangles.

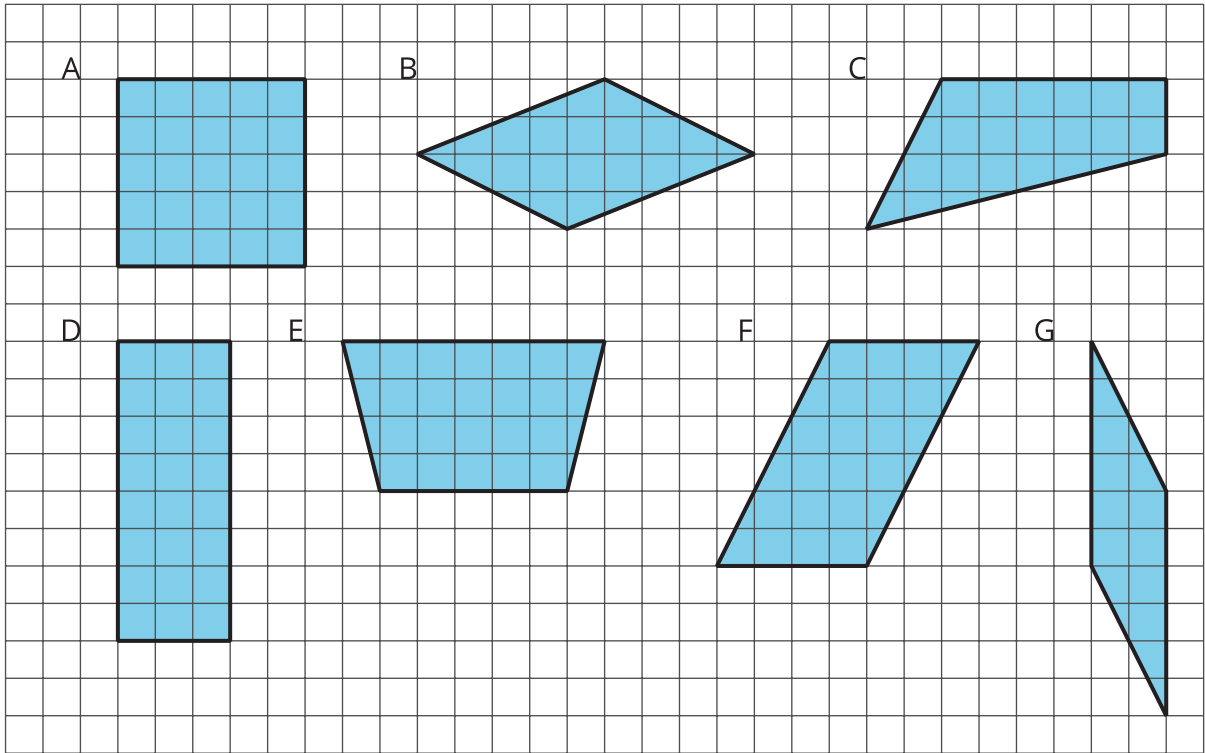
Supports accessibility for: Fine Motor Skills, Visual-Spatial Processing

Student Task Statement

 Two shapes are identical if they match up exactly when placed one on top of the other.



1. Draw *one* line to decompose each shape into two identical triangles, if possible. Use a straightedge to draw your line.



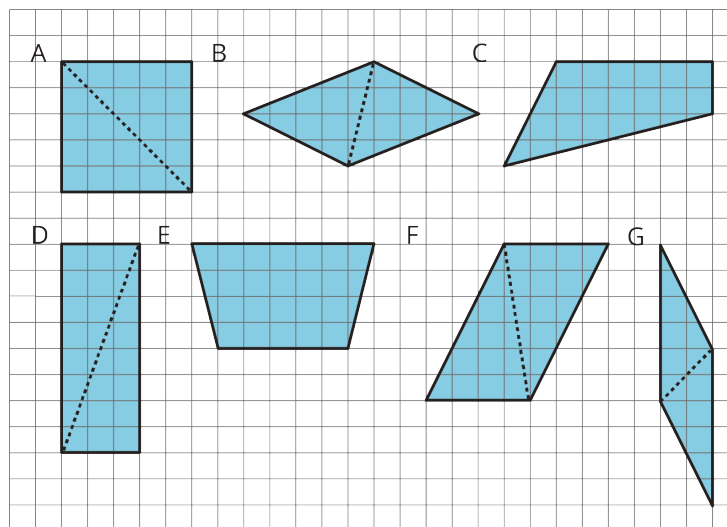
2. Which quadrilaterals can be decomposed into two identical triangles?

Pause here for a small-group discussion.

3. Study the quadrilaterals that can, in fact, be decomposed into two identical triangles. What do you notice about them? Write a couple of observations about what these quadrilaterals have in common.

Student Response

1. Sample response:



2. Quadrilaterals A, B, D, F, and G can be decomposed into two identical triangles.
3. Sample responses:
 - They have two pairs of parallel sides and each pair has equal length.
 - They are all parallelograms.
 - The triangles are formed by drawing a diagonal connecting opposite vertices.
 - Some triangles are right triangles, some are acute, and some are obtuse.
 - For some quadrilaterals, there is more than one way to decompose it into two identical triangles.

Building on Student Thinking

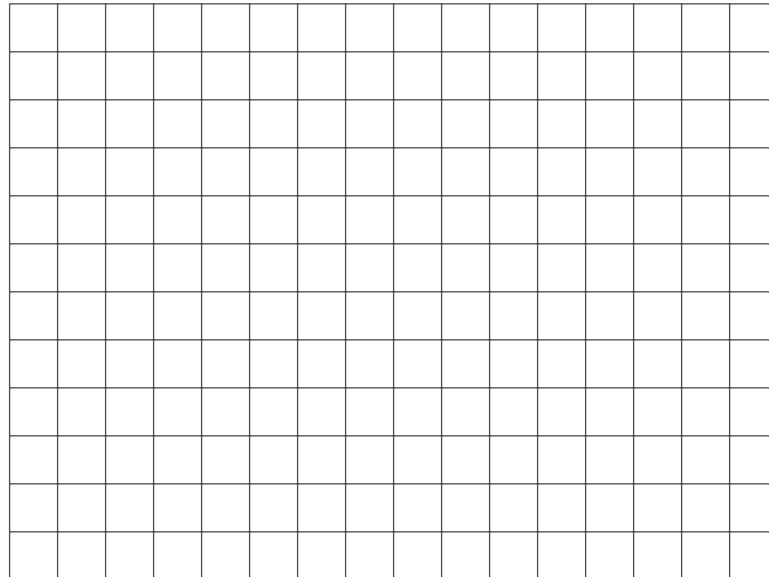
It may not occur to students to rotate triangles to check congruence. If so, tell students that we still consider two triangles identical even when one needs to be rotated to match the other.

Students may incorrectly generalize that Figure E can be decomposed into two identical triangles because it shares some common characteristics with Figures A, B, and D: It has two sides that are parallel, two sides that are the same length, and two pairs of equal-size angles. Remind students to use the tools at their disposal to verify their thinking.



Are You Ready for More?

On the grid, draw some other types of quadrilaterals that are not already shown. Try to decompose them into two identical triangles. Can you do it?



Come up with a rule about what must be true about a quadrilateral for it to be decomposed into two identical triangles.

Extension Student Response

Answers vary.



Activity Synthesis

The discussion should serve two goals: to highlight how quadrilaterals can be decomposed into triangles and to help students make generalizations about the types of quadrilaterals that can be decomposed into two identical triangles. Consider these questions:

- “How did you decompose the quadrilaterals into two triangles?” (Connect opposite vertices by drawing a diagonal.)
- “Did the strategy of drawing a diagonal work for decomposing all quadrilaterals into two triangles?” (Yes.) “Are all of the resulting triangles identical?” (No.)
- “What do A, B, and D have that C and E do not?” (A, B, and D have two pairs of parallel sides that are of equal lengths. They are parallelograms.)
- “What type of quadrilateral can always be decomposed into two congruent triangles?” (a parallelogram)
- “What features must a quadrilateral have for it to be decomposable into two identical triangles?” (two pairs of congruent sides and one pair of congruent angles)

If students point out that a kite also can be decomposed into two congruent triangles, ask “What features do kites and parallelograms have in common?”

If time permits, discuss how students verified the congruence of the two triangles.

- “How did you check if the triangles are identical? Did you simply stack the traced triangle or did you do something more specific?” (They may notice that it is necessary to rotate one triangle—or to reflect one triangle twice—before the triangles could be matched up.)
- “Did anyone use another way to check for congruence?” (Students may also think in terms of the parts or composition of each triangle. For example, they might say that both triangles have all the same side lengths and have a right angle.)

6.3

A Tale of Two Triangles (Part 2)

🕒 15 min

Activity Narrative

There is a digital version of this activity.

In this activity, students compose quadrilaterals using pairs of identical triangles. Previously, students saw that a triangle can be seen as half of a familiar quadrilateral. This activity prompts them to think the other way—to see that two identical triangles of any kind can always be joined to produce a parallelogram. Both explorations prepare students to make connections between the area of a triangle and that of a parallelogram in a subsequent lesson.

A key understanding to uncover here is that two identical copies of a triangle can be joined along any corresponding side to compose a parallelogram. This means more than one parallelogram can be formed by the same pair of triangles.

As students work, look for different compositions of the same pair of triangles. Select students who use different approaches to share.

When manipulating the cutouts, students are likely to notice that right triangles can be composed into rectangles (and sometimes squares) and that non-right triangles produce parallelograms that are not rectangles.

As before, students make generalizations here that they don't yet have the tools to justify. This is appropriate at this stage. Later in their study of mathematics, they will learn to verify what they now take as facts.



In the digital version of the activity, students use an applet to compose pairs of triangles into quadrilaterals. The applet allows students to move and rotate the triangles.

Standards

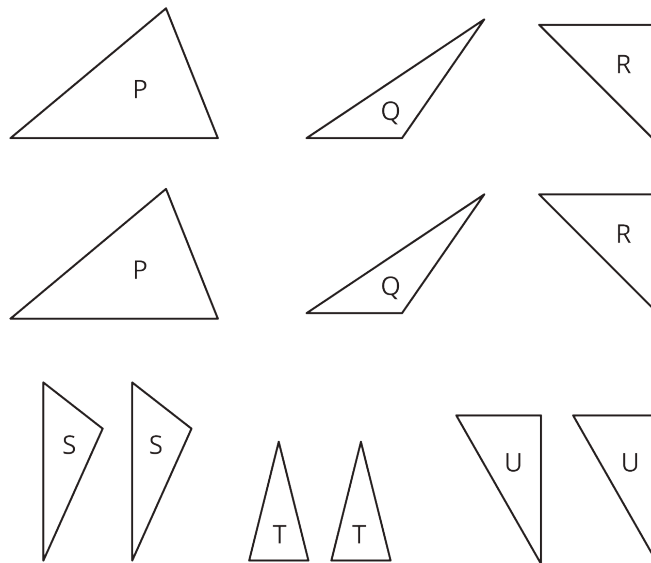
Building Toward 6.G.A.1

Launch

Keep students in the same groups. Give each group one set (6 pairs) of triangles labeled P, Q, R, S, T, and U from the blackline master and access to scissors if the triangles are not pre-cut. Instruct each group member to take 1–2 pairs of triangles.

Remind students that in the previous activity they saw that certain types of quadrilaterals can be decomposed into two identical triangles. Explain that they will now see if it is possible to compose quadrilaterals out of two identical triangles and, if so, to find out what types of quadrilaterals would result.

Give students 1–2 minutes of quiet work time for the first question. Then, give them 5 minutes to discuss their responses and answer the second question with their group.



Student Task Statement

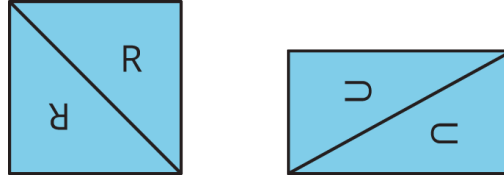
Your teacher will give your group several pairs of triangles. Each group member should take 1 or 2 pairs.

- Which pair(s) of triangles do you have?
 - Can each pair be composed into a rectangle? A parallelogram?
- Discuss with your group your responses to the first question. Then, complete each statement with *All*, *Some*, or *None*. Sketch 1 or 2 examples to illustrate each completed statement.
 - _____ of these pairs of identical triangles can be composed into a *rectangle*.
 - _____ of these pairs of identical triangles can be composed into a *parallelogram*.

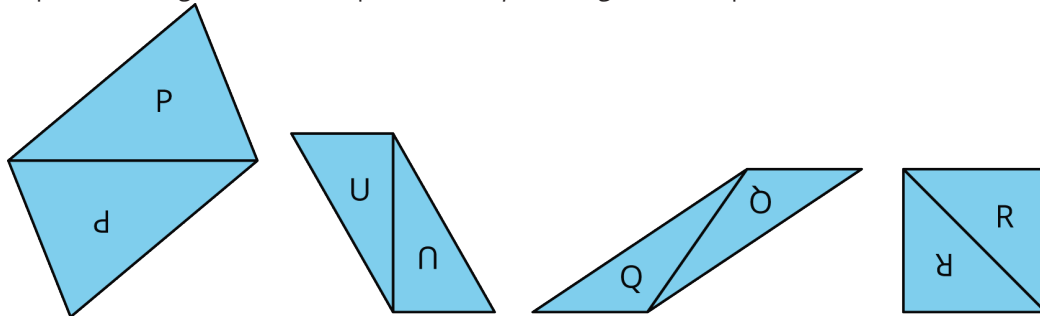


Student Response

- Answers vary.
 - Sample responses:
 - Yes for triangles R and U, no for the rest
 - Yes for all triangles
- Some of these pairs of triangles can be composed into a *rectangle*.



- All of these pairs of triangles can be composed into a *parallelogram*. Examples:



Building on Student Thinking

Students may draw incorrect conclusions if certain pieces of their triangles are turned over (to face down), or if it did not occur to them that the pieces could be moved. Ask them to try manipulating the pieces in different ways.

Seeing that two copies of a triangle can always be composed into a parallelogram, students might mistakenly conclude that any two copies of a triangle can *only* be composed into a parallelogram (that is, no other quadrilaterals can be formed from joining two identical triangles). Showing a counterexample may be a simple way to help students see that this is not the case.

Activity Synthesis

The focus of this discussion would be to clarify whether or not two copies of each triangle can be composed into a rectangle or a parallelogram, and to highlight the different ways in which two triangles could be composed into a parallelogram.

Ask a few students who composed different parallelograms from the same pair of triangles to share. Invite the class to notice how these students ended up with different parallelograms. To help them see that a triangle can be joined along any side of its copy to produce a parallelogram, ask questions such as:

- “Here is one way of composing Triangles S into a parallelogram. Did anyone else do it this way? Did anyone obtain a parallelogram in a different way?”
- “How many different parallelograms can be created with any two copies of a triangle? Why?” (3 ways, because there are 3 sides along which the triangles could be joined.)
- “What kinds of triangles can be used to compose a rectangle? How?” (Right triangles, by joining two copies along the



side opposite the right angle.)

- “What kinds of triangles can be used to compose a parallelogram? How?” (Any triangle, by joining two copies along any sides with the same length.)

Lesson Synthesis

Display and revisit representative works from the two main activities. Draw out key observations about the special connections between triangles and parallelograms.

First, students tried to decompose, or break apart, quadrilaterals into two identical triangles. Consider asking students:

- “What strategy allowed us to do that?” (Drawing a segment connecting opposite vertices.)
- “Which types of quadrilaterals could always be decomposed into two identical triangles?” (Parallelograms.)
- “Can quadrilaterals that are not parallelograms be decomposed into triangles?” (Yes, but the resulting triangles may not be identical.)

Then, students explored the relationship between triangles and quadrilaterals the other way around: by composing quadrilaterals from pairs of identical triangles. Consider asking students:

- “What types of quadrilaterals were you able to compose with a pair of identical triangles?” (Parallelograms—some of them are rectangles.)
- “Does it matter which triangles were used?” (No. Any two copies of a triangle could be composed into a parallelogram.)
- “Was there a particular side along which the two triangles must be joined to form a parallelogram?” (No. Any of the three sides could be used as long as the sides match.)

Emphasize that two identical copies of a triangle can be combined to make a parallelogram. This is true for any triangle. The reverse is also true: any parallelogram can be split into two identical triangles.

Tell students that in grade 8 they will acquire some tools to prove these observations. For now, they will take the special relationships between triangles and parallelograms as a fact and use those relationships to find the area of any triangle.

6.4 A Tale of Two Triangles (Part 3)

Cool-down

🕒 5 min

📖 Standards

Building Toward 6.G.A.1

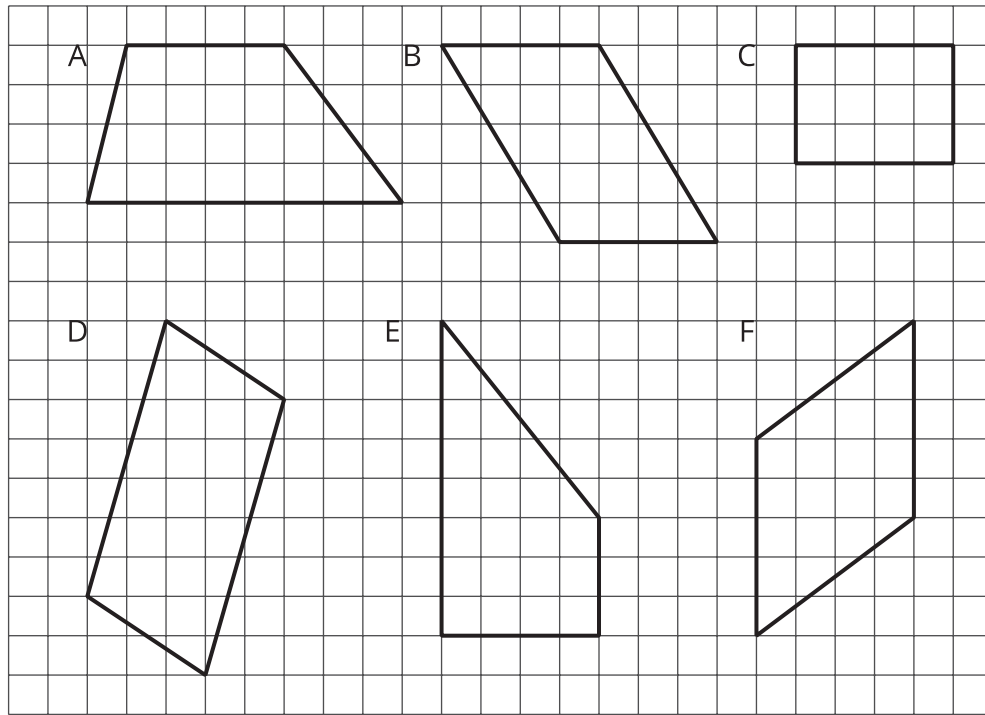
Launch

Give students access to their geometry toolkits if needed.

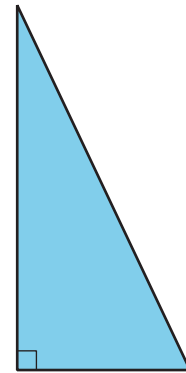
👤 Student Task Statement

1. Here are some quadrilaterals.





1.
 - a. Circle all quadrilaterals that you think can be decomposed into two identical triangles using only one line.
 - b. What characteristics do the quadrilaterals that you circled have in common?
2. Here is a right triangle. Show or briefly describe how two copies of it can be composed into a parallelogram.



Student Response

1.
 - a. Quadrilaterals B, C, D, and F should be circled.
 - b. They all have two pairs of parallel sides. They are all parallelograms.
2. Sample response: Joining two copies of the triangle along a side that is the same length (for instance, the shortest side of one and the shortest side of the other) would make a parallelogram. (Three parallelograms are possible, since there are three sides at which the triangles could be joined. One of the parallelograms is a rectangle.)

Responding to Student Thinking

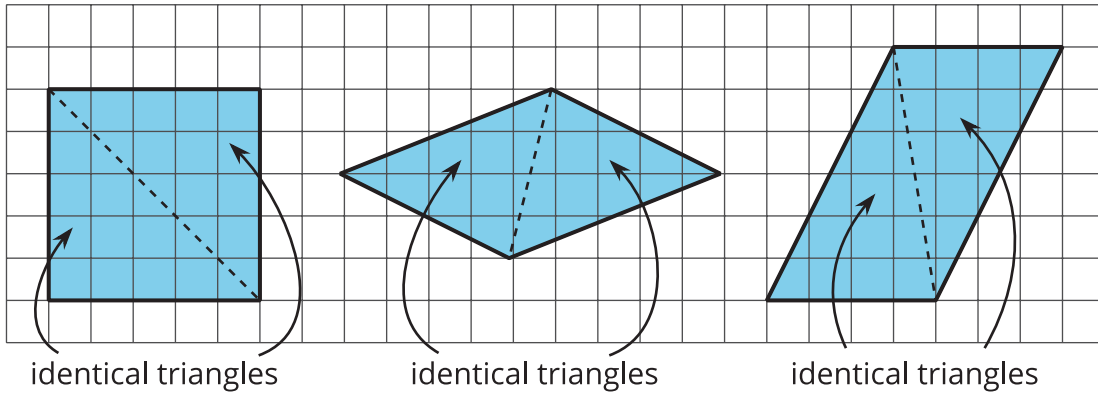
More Chances

Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

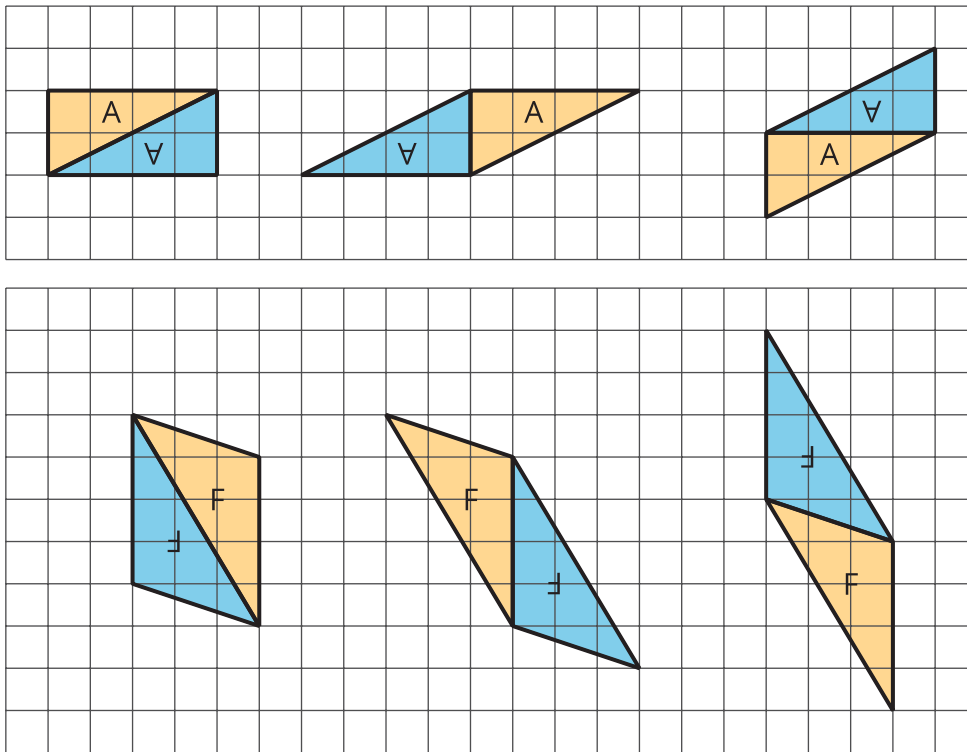


Lesson 6 Summary

A parallelogram can always be decomposed into two identical triangles by a segment that connects opposite vertices.



Going the other way around, two identical copies of a triangle can always be arranged to form a parallelogram, regardless of the type of triangle being used. To produce a parallelogram, we can join a triangle and its copy along any of the three sides that match, so the same pair of triangles can make different parallelograms. Here are examples of how two copies of both Triangle A and Triangle F can be composed into three different parallelograms.

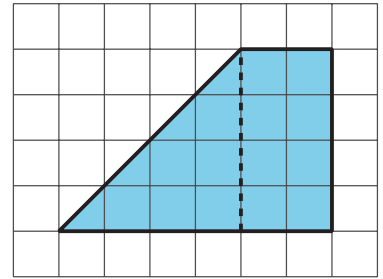


This special relationship between triangles and parallelograms can help us reason about the area of any triangle.

Lesson 6 Practice Problems

1 Student Task Statement

To decompose a quadrilateral into two identical shapes, Clare drew a dashed line as shown in the diagram.



- She said that the two resulting shapes have the same area. Do you agree? Explain your reasoning.
- Did Clare partition the figure into two identical shapes? Explain your reasoning.

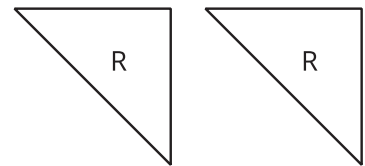
Solution

- Agree. Sample reasoning: The rectangle is 2 units by 4 units, so it has an area of 8 square units. The triangle is half of a 4-by-4 square, so its area is also 8 square units.
- No. Sample reasoning: Although the shapes have the same area, they are not identical shapes—one is a rectangle and the other is a triangle.

2 Student Task Statement

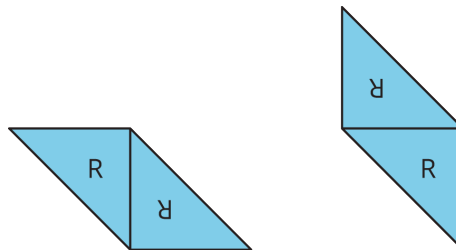
Triangle R is a right triangle. Can we use two copies of Triangle R to compose a parallelogram that is not a square?

If so, explain how or sketch a solution. If not, explain why not.



Solution

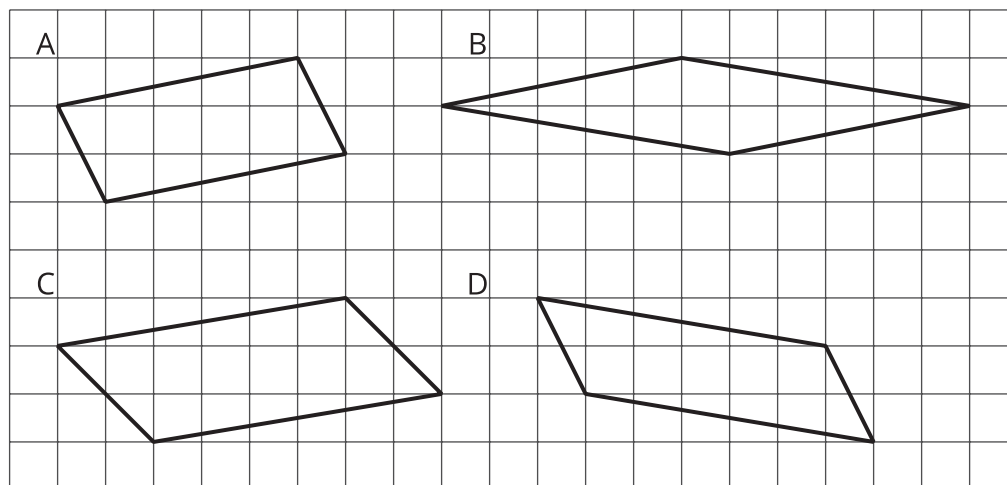
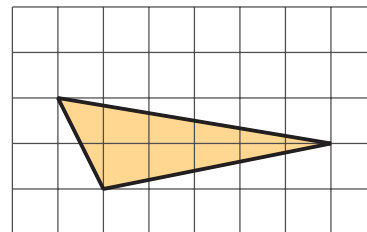
Yes. Sample reasoning: We can use two right triangles R to compose a parallelogram that is not a square by joining them along one of the shorter sides (the sides that make the right angle).



3 Student Task Statement

Two copies of this triangle are used to compose a parallelogram.

Which parallelogram *cannot* be a result of the composition? If you get stuck, consider using tracing paper.



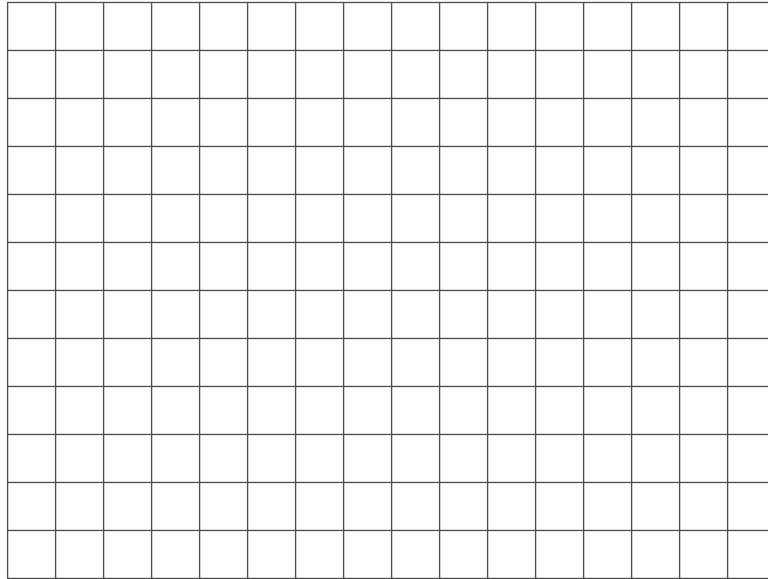
- A. A
- B. B
- C. C
- D. D

Solution

C

4 Student Task Statement

- a. On the grid, draw at least three different quadrilaterals that can each be decomposed into two identical triangles with a single cut (show the cut line). One or more of the quadrilaterals should have non-right angles.

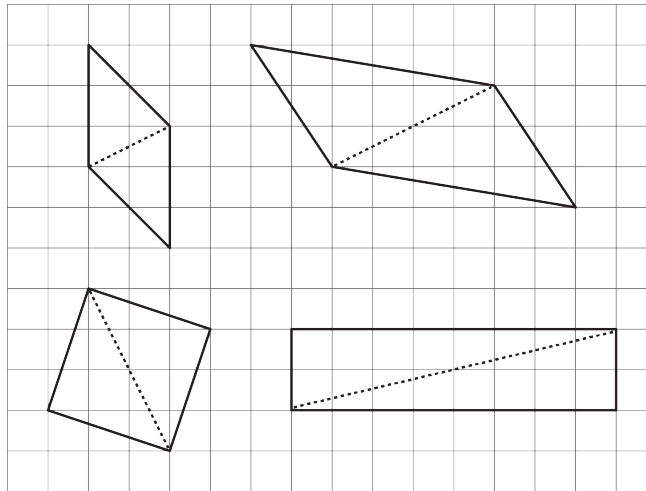


b. Identify the type of each quadrilateral.

Solution

Sample responses:

a.



b. The top two are parallelograms. The bottom left one is a square. The bottom right one is a rectangle. (All of them are parallelograms.)

5

from Unit 1, Lesson 5



Student Task Statement



a. A parallelogram has a base of 9 units and a corresponding height of $\frac{2}{3}$ units. What is its area?



- b. A parallelogram has a base of 9 units and an area of 12 square units. What is the corresponding height for that base?
- c. A parallelogram has an area of 7 square units. If the height that corresponds to a base is $\frac{1}{4}$ unit, what is the base?

Solution

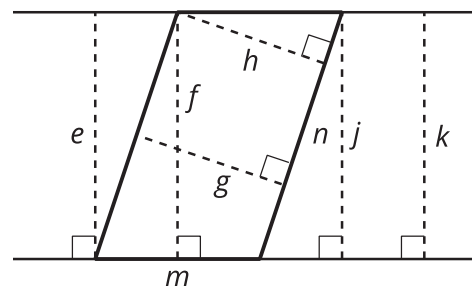
- a. $\frac{18}{3}$ square units (or equivalent)
- b. $\frac{12}{9}$ units (or equivalent)
- c. 28 units

6

from Unit 1, Lesson 5

Student Task Statement

Select **all** the segments that could represent the height if side n is the base.



- A. e
- B. f
- C. g
- D. h
- E. m
- F. n
- G. j
- H. k

Solution

C, D