Polynomial Identities (Part 2)

Let's explore some other identities.

9.1 Revisiting an Old Theorem

Instructions to make a right triangle:

- Choose two integers.
- Make one side length equal to the sum of the squares of the two integers.
- Make one side length equal to the difference of the squares of the two integers.
- Make one side length equal to twice the product of the two integers.

Follow these instructions to make a few different triangles. Do you think the instructions always produce a right triangle? Be prepared to explain your reasoning.

9.2 Theorems and Identities

Here are the instructions to make a right triangle from earlier:

- Choose two integers.
- Make one side length equal to the sum of the squares of the two integers.
- Make one side length equal to the difference of the squares of the two integers.
- Make one side length equal to twice the product of the two integers.

1. Using a and b for the two integers, write expressions for the three side lengths.

2. Why do these instructions make a right triangle when $a \neq b$?

9.3 Identifying Identities

Here is a list of equations. Circle all the equations that are identities. Be prepared to explain your reasoning.

1. $a = -a$
2. $a^2 + 2ab + b^2 = (a + b)^2$
3. $a^2 - 2ab + b^2 = (a - b)^2$
4. $a^2 - b^2 = (a - b)(a - b)$
5. $(a + b)(a^2 - ab + b^2) = a^3 - b^3$
6. $(a - b)^3 = a^3 - b^3 - 3ab(a + b)$
7. $a^2(a - b)^4 - b^2(a - b)^4 = (a - b)^5(a + b)$

9.4 Egyptian Fractions

In Ancient Egypt, all non-unit fractions were represented as a sum of distinct unit fractions. For example, $\frac{4}{9}$ would have been written as $\frac{1}{3}+\frac{1}{9}$ (and not as $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}$ or any other form with the same unit fraction used more than once). Let's look at some different ways we can rewrite $\frac{2}{15}$ as the sum of distinct unit fractions.

1. Use the formula $\frac{2}{d}=\frac{1}{d}+\frac{1}{2d}+\frac{1}{3d}+\frac{1}{6d}$ to rewrite the fraction $\frac{2}{15}$, then show that this formula is an identity.

2. Another way to rewrite fractions of the form $\frac{2}{d}$ is given by the identity $\frac{2}{d}=\frac{1}{d}+\frac{1}{d+1}+\frac{1}{d(d+1)}$. Use it to rewrite the fraction $\frac{2}{15}$, then show that it is an identity.

Are you ready for more?

For fractions of the form $\frac{2}{pq}$, that is, fractions with a denominator that is the product of two positive integers, the following formula can also be used: $\frac{2}{pq} = \frac{1}{pr} + \frac{1}{qr}$, where $r = \frac{p+q}{2}$. Use it to rewrite the fraction $\frac{2}{45}$, then show that it is an identity.

Lesson 9 Summary

Sometimes we can think something is an identity when it actually isn't. Consider the following equations that are sometimes mistaken as identities:

$$(a+b)^2 = a^2 + b^2$$

$$(a-b)^2 = a^2 - b^2$$

Both of these are true for some very specific values of a and b (for example, when either a or b is 0), but they are not true for most values of a and b, for example $a = 2$ and $b = 1$ (try it!). The actual identities associated with the expressions on the left side are $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.

Are polynomials the only types of expressions you can find in identities? Not at all! Here is an identity that shows a relationship between rational expressions:

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

We can show that this identity is true by adding the terms in the expression on the right using a common denominator:

$$\begin{aligned} \tfrac{1}{x+1} + \tfrac{1}{x(x+1)} &= \tfrac{1}{x+1} \cdot \tfrac{x}{x} + \tfrac{1}{x(x+1)} \\ &= \tfrac{x}{x(x+1)} + \tfrac{1}{x(x+1)} \\ &= \tfrac{x+1}{x(x+1)} \\ &= \tfrac{1}{x} \end{aligned}$$

An important difference from polynomial identities is that identities involving rational expressions could have a few exceptional values of x for which they are not true because the rational expressions on one side or the other are not defined. For example, the identity above is true for all values of x except $x = 0$ and $x = \text{-}1$.