Unit 2, Lesson 13

Weighted Averages in a Triangle

Goals

- Calculate the intersection point of a triangle's medians.
- Prove that the medians of a triangle meet at a point.

Learning Targets

- I can determine the point where the medians of a triangle intersect.

Lesson Narrative

The goal of this lesson is to use segment partitioning logic to prove that the medians of a triangle intersect in a single point.

Students begin by finding midpoints of the sides of triangles, and they learn that **medians** are segments that connect a triangle vertex to the midpoint of the opposite side. Next, they find points that partition the medians of a triangle in a $2:1$ ratio, observing in the process that the 3 medians of the triangle intersect in a single point. Finally, students work in groups to construct a viable argument that proves the medians of *any* triangle intersect at that $2:1$ partition point (MP3). While the intersection point of the medians isn't given the formal name *centroid* in this lesson, it may be helpful to use that term.

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. We recommend making technology available.

Standards

Building On	**HSG-CO.C.10, HSG-GPE.B.6**
Addressing	**HSG-CO.C.10, HSG-GPE.B.4, HSG-GPE.B.6, HSN-Q.A.1**
Building Toward	**HSG-CO.C.10**

Instructional Routines

- Draw It
- MLR1: Stronger and Clearer Each Time
- Notice and Wonder

Required Materials

Materials to Gather

- Index cards: Lesson
- Rulers: Lesson, Activity 2
- Scissors: Lesson

Required Preparation

Activity 2:

Each student will need a ruler to measure and as a straight edge.

Lesson:

Students will use rulers as a straightedge and for taking measurements in both the "Triangle Medians" activity and the *Lesson Synthesis*.

The index cards and scissors are for use in the *Lesson Synthesis*.

Student Facing Learning Goals

Let's partition special line segments in triangles.

13.1 Triangle Midpoints

Warm-up

5 min

Activity Narrative

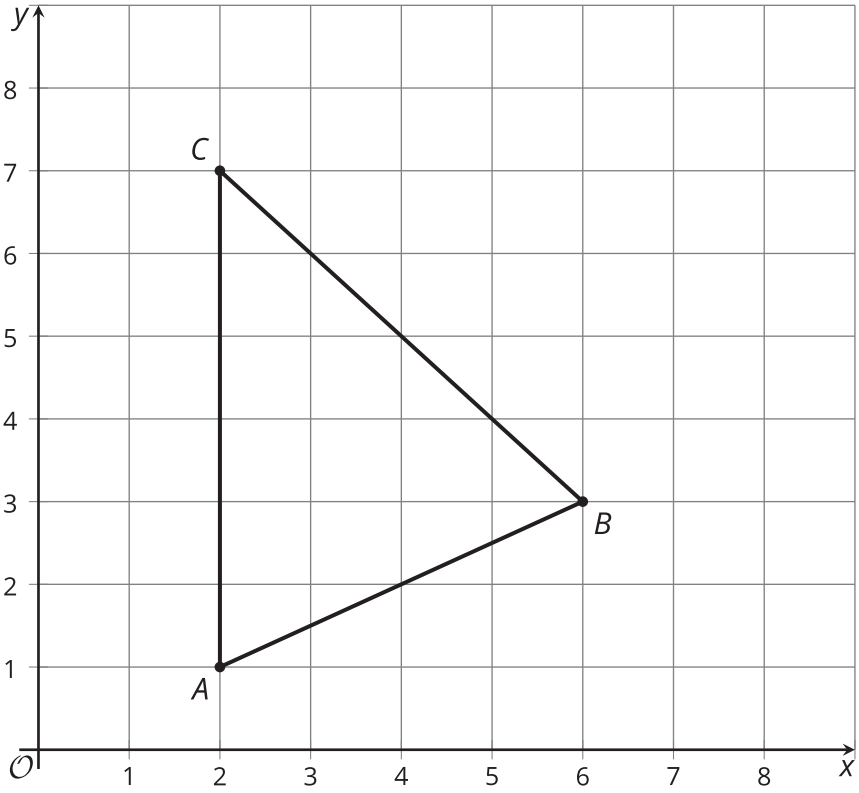
In this activity, students find the midpoint of each side of a triangle. This allows for practice using an average to find the midpoint and builds the basis for the diagram in the next activity. Monitor for students who use different strategies to find the midpoint, such as counting, averaging x-coordinates and then y-coordinates as in the traditional midpoint formula, and using the notation $\frac{1}{2}A + \frac{1}{2}B$.

Standards

Addressing **HSG-GPE.B.6**

Student Task Statement

Triangle ABC is graphed.



Find the midpoint of each side of this triangle.

Student Response

$(2, 4), (4, 2), (4, 5)$

Activity Synthesis

Invite previously selected students to share their strategies.

If not mentioned, ask students to calculate $\frac{1}{2}A + \frac{1}{2}B$ and then to compare it to their diagram.

13.2 Triangle Medians

15 min

Activity Narrative

Students examine the medians of a particular triangle. After noticing that the 3 segments appear to coincide at a point partitioning the median in a $2:1$ ratio, students prove that is the case for this particular triangle. They will generalize this idea in the next activity.

Making dynamic geometry software available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Building On	**HSG-GPE.B.6**
Building Toward	**HSG-CO.C.10**

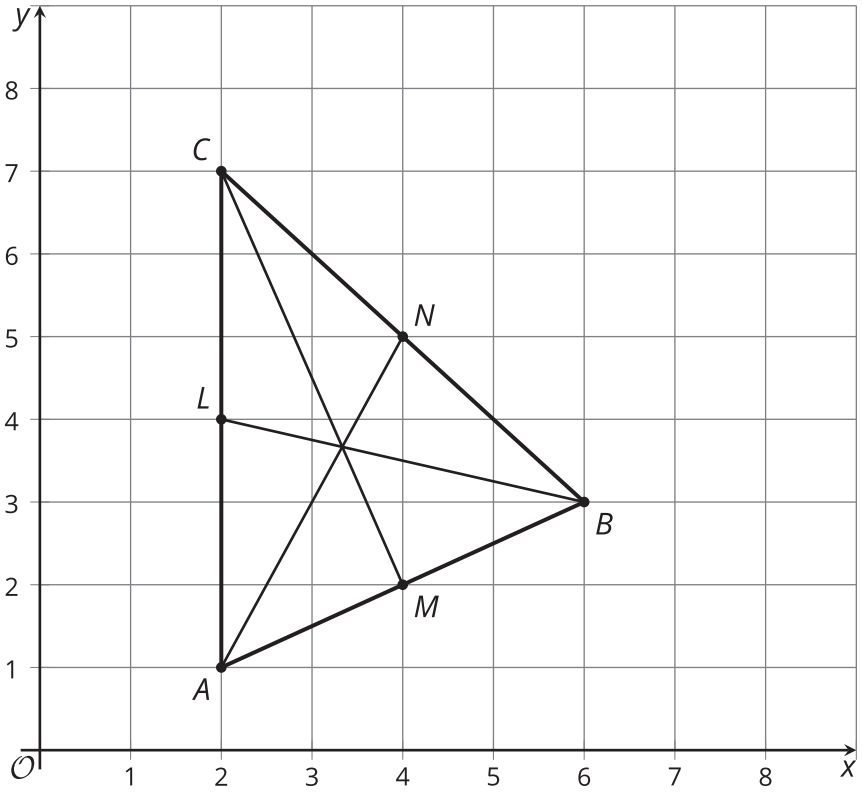
Instructional Routines

- Draw It
- Notice and Wonder

Launch

Tell students that a **median** of a triangle is a line segment connecting a vertex to the midpoint of the opposite side. Ask students how many medians can be drawn in any triangle (3).

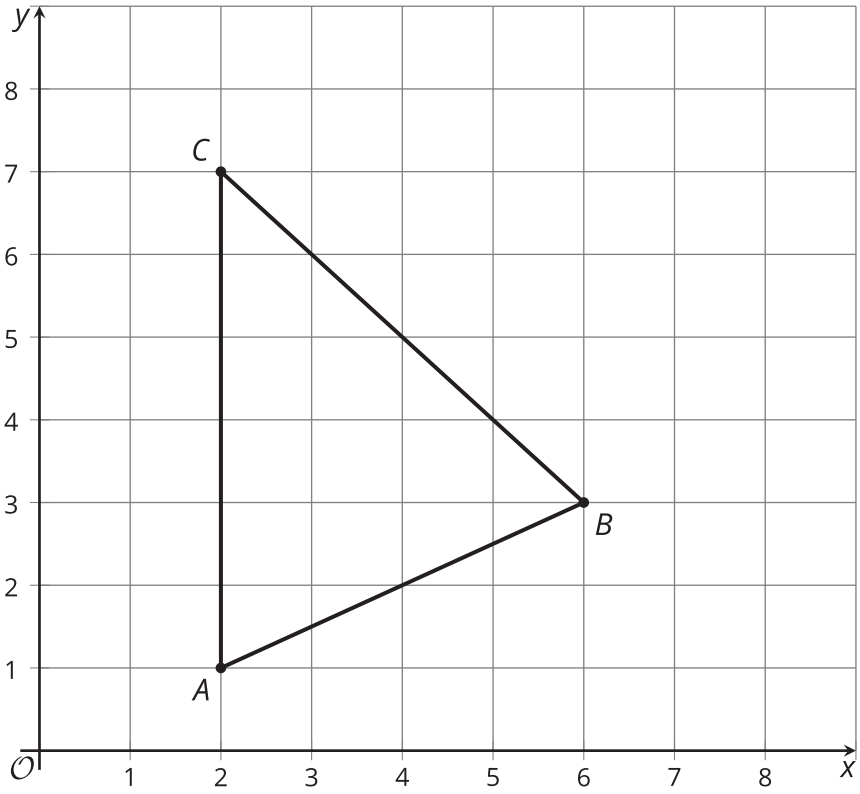
Now ask students to draw the 3 medians of the triangle in the *Warm-up*, by using a straightedge to connect the midpoint of each side to the opposite vertex. What do you notice? (The medians seem to intersect at one point.) Instruct students to label the diagram as shown here.



Access for Students with Disabilities

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide materials to allow students to physically compare the parts of each median. *Supports accessibility for: Conceptual Processing*

Student Task Statement



Your teacher will tell you how to draw and label the **medians** of this triangle.

1. After the medians are drawn and labeled, measure all 6 segments inside the triangle using centimeters. What is the ratio of the 2 parts of each median?
2. Find the coordinates of the point that partitions segment AN in a $2 : 1$ ratio.
3. Find the coordinates of the point that partitions segment BL in a $2 : 1$ ratio.
4. Find the coordinates of the point that partitions segment CM in a $2 : 1$ ratio.

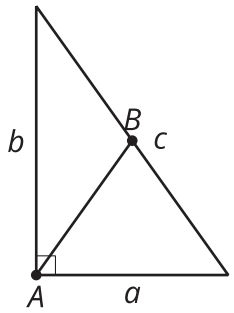
Student Response

1. $2 : 1$
2. $\left(\frac{10}{3}, \frac{11}{3}\right)$ (or equivalent)
3. $\left(\frac{10}{3}, \frac{11}{3}\right)$ (or equivalent)
4. $\left(\frac{10}{3}, \frac{11}{3}\right)$ (or equivalent)

Are You Ready for More?

In the image, AB is a median.

Find the length of AB in terms of a, b, and c.



Extension Student Response

$$\frac{\sqrt{(a^2 + b^2)}}{2}$$

Activity Synthesis

The purpose of this discussion is for students to describe their strategies for partitioning medians of a particular triangle. Invite a student to share a strategy for finding the point that partitions one of the medians.

Display the image of the triangle with its medians. Give students 1 minute of quiet think time, and ask them to be prepared to share at least one thing that they notice and one thing that they wonder. Record and display their responses without editing or commentary. If possible, record the relevant reasoning on or near the image.

Things students may notice:

- The medians intersect at a single point.
- The ratio is the same for each median.
- The point of intersection partitions each median in a $2 : 1$ ratio.

Things students may wonder:

- Do the medians of a triangle always intersect in a single point?
- Does an intersection point always partition the medians in the same ratio?
- Is there something special about the $2 : 1$ ratio, or could it be any ratio for a different triangle?

If the medians intersecting in a single point does not come up during the conversation, ask students to discuss this idea.

13.3 Any Triangle's Medians

15 min

Activity Narrative

Students generalize their observations and explanation from the previous activity to prove that the medians of a triangle always coincide. They build from the wonderings or conjectures from earlier activities to show that the point that partitions each triangle median in a $2 : 1$ ratio is the same point and, therefore, is a point of intersection.

As students generalize thinking from specific triangles to the medians of any triangle, they construct viable arguments (MP3).

Standards

Addressing **HSG-CO.C.10, HSG-GPE.B.4, HSN-Q.A.1**

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Arrange students in groups of 3–4. Give students 2 minutes of work time to read and interpret the beginning of the problem, monitoring for groups that draw a diagram. Then, pull the class back together briefly, inviting a group to share its diagram.

Access for Students with Disabilities

Engagement: Develop Effort and Persistence. Provide reminders or checklists that focus on increasing the length of on-task orientation in the face of distractions. For example, provide a checklist of the responsibilities of each group member, with pre-assigned sides and medians.
Supports accessibility for: Attention, Social-Emotional Functioning

Student Task Statement

The goal is to prove that the medians of any triangle intersect at a point. Suppose the vertices of a triangle are $(0, 0)$, $(w, 0)$, and (a, b).

1. Each student in the group should choose 1 side of the triangle. If your group has four people, two can work together. Write an expression for the midpoint of the side you chose.
2. Each student in the group should choose a median. Write an expression for the point that partitions each median in a $2 : 1$ ratio from the vertex to the midpoint of the opposite side.
3. Compare the coordinates of the point you found to those of your groupmates. What do you notice?
4. Explain how these steps prove that the 3 medians of any triangle intersect at a single point.

Student Response

1. The three midpoints are given by $\left(\frac{a}{2}, \frac{b}{2}\right)$, $\left(\frac{a+w}{2}, \frac{b}{2}\right)$, and $\left(\frac{w}{2}, 0\right)$.
2. $\left(\frac{a+w}{3}, \frac{b}{3}\right)$
3. All 3 points were the same.
4. Sample response: Given any triangle, use a sequence of rigid motions to take it to an image where the coordinates of its vertices are $(0, 0)$, $(w, 0)$, and (a, b). The medians of this triangle are defined by connecting the midpoint of each side to the opposite vertex. These midpoints are distinct and not collinear, so the medians do not coincide. So, if there is 1 point that is common to all 3 medians, they must intersect at that point. Hypothesize that this point is the one that partitions each median in a $2 : 1$ ratio from the vertex to the midpoint. Calculations show that this partition point is $\left(\frac{a+w}{3}, \frac{b}{3}\right)$ for all 3 medians. Therefore, the medians intersect at a single point.

Building on Student Thinking

If students struggle to find the point that partitions the median in a $2 : 1$ ratio, remind them of the notation they used in

earlier activities: $\frac{1}{3}A + \frac{2}{3}B$. How does that apply to this problem?

Activity Synthesis

The goal of the discussion is to ensure that students have incorporated all the necessary elements in their proof. Here are some questions for discussion:

- "What can we conclude if we know that a single point is on 3 different lines?" (It means that either the lines coincide or the lines intersect at that point.)
- "How do we know that the medians don't coincide?" (The midpoints are all different because the sides of the triangles don't coincide.)
- "Why did we test a $2:1$ ratio?" (That's the ratio we measured for the triangle in the *Warm-up*, so we thought it might apply to all triangles.)
- "Does this proof apply to a triangle that isn't positioned like the one in the diagram?" (Yes. We can use a sequence of rigid motions to take any triangle to an image in which a vertex is at $(0, 0)$ and in which one side coincides with the x-axis.)

Give students 1–2 minutes to revise their proofs with any additional information needed.

Access for English Language Learners

MLR1 Stronger and Clearer Each Time. Before the whole-class discussion, give students time to meet with 2–3 partners to share and get feedback on their first draft response to how the steps prove that the three medians of any triangle intersect at a single point. Invite listeners to ask questions and give feedback that will help clarify and strengthen their partners' ideas and writing. Give students 3–5 minutes to revise their first draft based on the feedback they receive.
Advances: Writing, Speaking, Listening

Lesson Synthesis

Invite students to draw a triangle on an index card and to locate the point where the medians all intersect. Students may use folding or measuring to find midpoints. Is it possible to find the intersection point without drawing all 3 medians? (Yes. Draw 2 and find the intersection point, or draw 1 and find the point that partitions the segment in a $2:1$ ratio.)

It turns out that this point is the balancing point of a triangle. Challenge students to cut out their triangles and balance them on the eraser of a pencil. If they were really precise, the triangles will balance on the tip of the pencil as well. Consider collecting all of these triangles and making a mobile to decorate the classroom.

13.4 Intersection Point

Cool-down

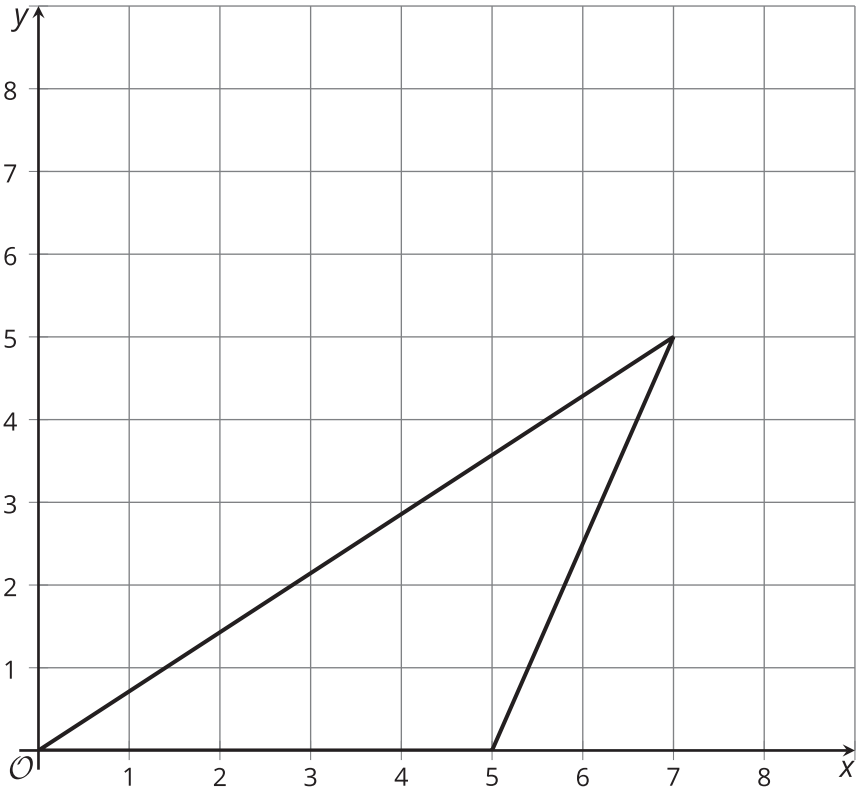
5 min

Standards

Building On **HSG-CO.C.10**

Student Task Statement

Where do the medians of this triangle intersect?



Student Response

$\left(4, \frac{5}{3}\right)$

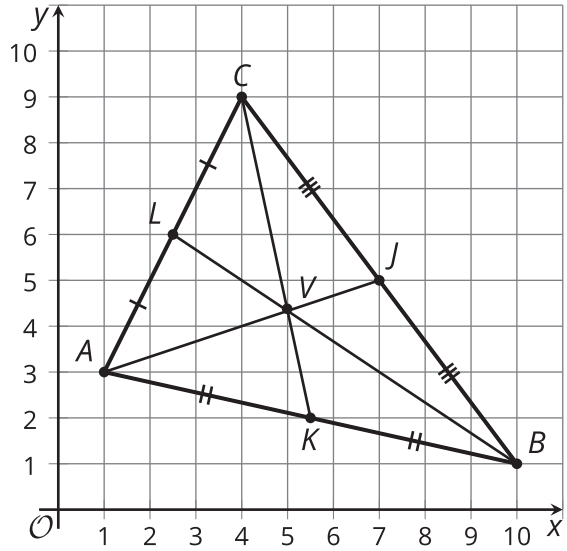
Responding to Student Thinking

Press Pause
By this point in the unit, there should be some student mastery of finding the medians of a triangle. If most students struggle, make time to revisit related work in the lessons referred to here. See the Course Guide for ideas to help students re-engage with earlier work.

Integrated Math 2, Unit 2, Lesson 12 Weighted Averages
Integrated Math 2, Unit 2, Lesson 13 Weighted Averages in a Triangle

Lesson 13 Summary

Here is a triangle with its medians drawn in. A **median** is a line segment drawn from a vertex of a triangle to the midpoint of the opposite side. Triangles have 3 medians, with 1 for each vertex.



Notice that the medians intersect at 1 point. This point is always $\frac{2}{3}$ of the distance from a vertex to the opposite midpoint. Another way to say this is that the point of intersection, V, partitions segments AJ, BL, and CK so that the ratios $AV : VJ$, $BV : VL$, and $CV : VK$ are all $2 : 1$.

We can prove this by working with a general triangle that can represent any triangle. Since any triangle can be transformed so that 1 vertex is on the origin and 1 side lies on the x-axis, we can say that our general triangle has vertices $(0, 0)$, $(w, 0)$, and (a, b). Through careful calculation, we can show that all 3 medians go through the point $\left(\frac{a+w}{3}, \frac{b}{3}\right)$. Therefore, the medians intersect at this point, which partitions each median in a $2 : 1$ ratio from the vertex to the opposite side's midpoint.

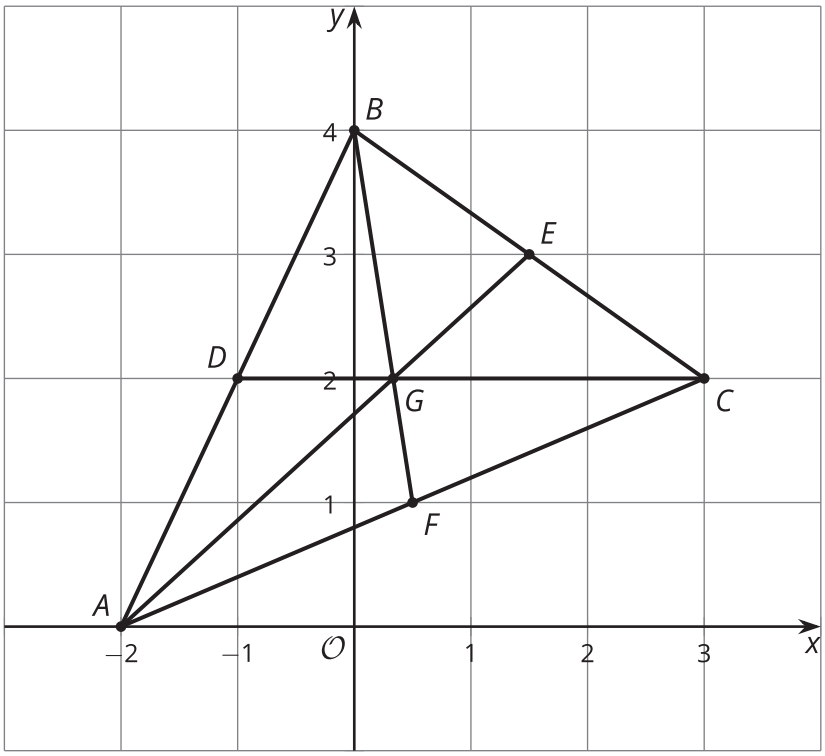
Glossary

- median (geometry)

Lesson 13 Practice Problems

1 Student Task Statement

Triangle ABC and its medians are shown.



Select **all** statements that are true.

A. The medians intersect at $\left(\frac{1}{3}, 2\right)$.

B. The medians and altitudes are the same for this triangle.

C. An equation for median AE is $y = \frac{6}{7}(x + 2)$.

D. Point G is $\frac{2}{3}$ of the way from A to E.

E. Median BF is congruent to median CD.

Solution

A, C, D

2 Student Task Statement

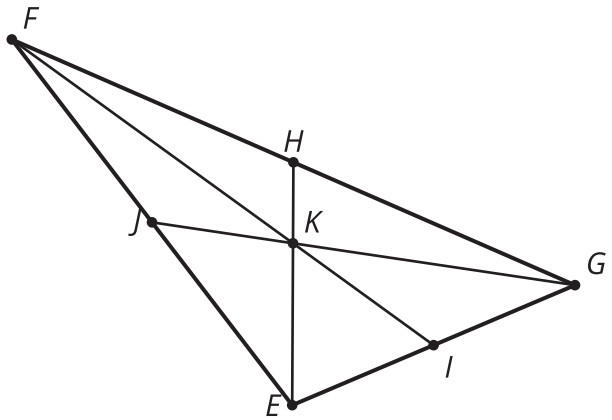
Triangle ABC has vertices at $(-2, 0)$, $(-1, 6)$, and $(6, 0)$. What is the point of intersection of the triangle's medians?

Solution

$(1, 2)$

3 Student Task Statement

Triangle EFG and its medians are shown.



Match each pair of segments with the ratios of their lengths.

A. $GK : KJ$	1. $1 : 1$
B. $GH : HF$	2. $1 : 2$
C. $HK : KE$	3. $2 : 1$

Solution

- A matches 3
- B matches 1
- C matches 2

4 from Unit 2, Lesson 12

Student Task Statement

Given $A(-3, 2)$ and $B(7, -10)$, find the point that partitions segment AB in a $1 : 4$ ratio.

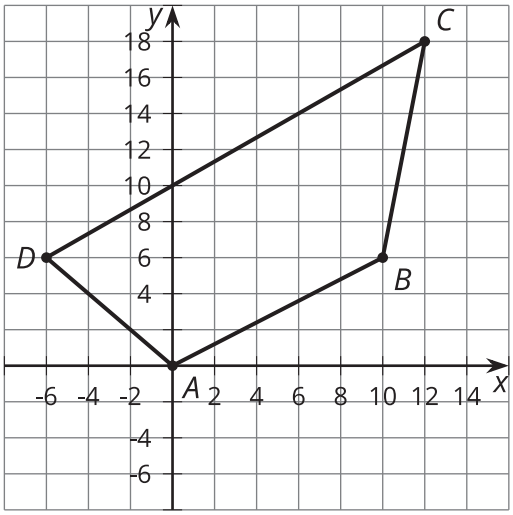
Solution

$(-1, -0.4)$

5 from Unit 2, Lesson 12

Student Task Statement

Graph the image of quadrilateral $ABCD$ under a dilation using center A and scale factor $\frac{1}{3}$.



Solution

