Solving Exponential Equations

Let's solve equations using logarithms.

14.1 A Valid Solution?

Here is a solution to the equation $5 \cdot e^{3a} = 90$.

$$\begin{aligned} 5 \cdot e^{3a} &= 90 \\ e^{3a} &= 18 \\ 3a &= \log_e 18 \\ a &= \frac{\log_e 18}{3} \end{aligned}$$

Explain what happened in each step.

14.2 Natural Logarithm

1. Complete the table with equivalent equations. The first row is completed for you.

	exponential form	logarithmic form
	$e^6 \approx 403.43$	$\ln(403.43) \approx 6$
a.	$e^0 = 1$	
b.	$e^1 = e$	
c.	$e^{-1} = \frac{1}{e}$	
d.		$\ln \frac{1}{e^2} = -2$
e.	$e^x = 10$	

2. Solve each equation by expressing the solution using $\ln$ notation. Then, find the approximate value of the solution using the "ln" button on a calculator.

   a. $e^m = 20$

   b. $e^n = 30$

   c. $e^p = 7.5$

14.3 Solving Exponential Equations

Without using a calculator, solve each equation. It is expected that some solutions will be expressed using log notation. Be prepared to explain your reasoning.

1. $10^x = 10{,}000$
2. $5 \cdot 10^x = 500$
3. $10^{(x+3)} = 10{,}000$
4. $10^{2x} = 10{,}000$
5. $10^x = 315$
6. $2 \cdot 10^x = 800$
7. $10^{(1.2x)} = 4{,}000$
8. $7 \cdot 10^{(0.5x)} = 70$
9. $2 \cdot e^x = 16$
10. $10 \cdot e^{3x} = 250$

Are you ready for more?

Assume that a and b are positive values. Use your understanding of what logarithms mean to find these values. Explain or show your reasoning.

1. $\log_a(a^b)$

2. $a^{\log_a(b)}$

Lesson 14 Summary

So far we have solved exponential equations by

- Finding whole number powers of the base (for example, the solution to $10^x = 100$ is $x = 2$, and the solution to $10^y = 1{,}000$ is $y = 3$).
- Estimation (for example, the solution of $10^x = 300$ is between 2 and 3 because 300 is between 100 and 1,000).
- Using a logarithm and approximating its value on a calculator (for example, the solution of $10^x = 300$ is $\log 300 \approx 2.48$).

Sometimes solving exponential equations takes additional reasoning. Here are a couple of examples.

$$\begin{aligned} 5 \cdot 10^x &= 45 \\ 10^x &= 9 \\ x &= \log 9 \end{aligned}$$

$$\begin{aligned} 10^{(0.2t)} &= 1{,}000 \\ 10^{(0.2t)} &= 10^3 \\ 0.2t &= 3 \\ t &= \frac{3}{0.2} \\ t &= 15 \end{aligned}$$

In the first example, the power of 10 is multiplied by 5, so to find the value of x that makes this equation true, each side is divided by 5. From there, the equation is rewritten as a logarithm, giving an exact value for x.

In the second example, the expressions on each side of the equation are rewritten as powers of 10: $10^{(0.2t)} = 10^3$. This means that the exponent $0.2t$ on one side and the 3 on the other side must be equal, and leads to an expression to solve where we don't need to use a logarithm.

How do we solve an exponential equation with base e, such as $e^x = 5$? We can express the solution using the **natural logarithm**, the logarithm for base e. Natural logarithm is written as $\ln$, or sometimes as $\log_e$. Just like the equation $10^2 = 100$ can be rewritten, in logarithmic form, as $\log_{10} 100 = 2$ or $\log 100 = 2$, the equation $e^0 = 1$ and be rewritten as $\ln 1 = 0$. Similarly, $e^{-2} = \frac{1}{e^2}$ can be rewritten as $\ln \frac{1}{e^2} = -2$.

All this means that we can solve $e^x = 5$ by rewriting the equation as $x = \ln 5$. This says that x is the exponent to which base e is raised to equal 5.

To estimate the size of $\ln 5$, remember that e is about 2.7. Because 5 is greater than e^1, this means that $\ln 5$ is greater than 1. e^2 is about $(2.7)^2$, or 7.3. Because 5 is less than e^2, this means that $\ln 5$ is less than 2. This suggests that $\ln 5$ is between 1 and 2. Using a calculator we can check that $\ln 5 \approx 1.61$.