Unit 5, Lesson 9

# Comparing Graphs

## Goals

- Compare key features of graphs of functions, and interpret them in context.
- Interpret equations of the form $f(x) = g(x)$ in context, and recognize that the solutions to such an equation are the $x$-coordinates of the points where the graphs of $f$ and $g$ intersect.
- Interpret statements about two or more functions written in function notation.

## Learning Targets

- I can compare the features of graphs of functions and explain what they mean in the situations represented.
- I can make sense of an equation of the form $f(x) = g(x)$ in terms of a situation and a graph, and know how to find the solutions.
- I can make sense of statements about two or more functions when they are written in function notation.

## Lesson Narrative

In this lesson, students deepen their understanding of functions by comparing representations of several functions relating the same pair of quantities. They analyze two or more graphs simultaneously, interpreting their relative features and their average rates of change in context.

Students also study comparative statements in function notation, such as $A(x) = B(x)$ or $B(10) > A(10)$, and explain them in terms of changes in population, changes in the trends of phone ownership, and the popularity of different television shows. Particular attention is given to any intersection points between graphs as places where two functions share an output value for the same input value.

Making comparisons involves looking beyond individual pieces of information. To accurately relate the information from multiple representations requires careful and precise use of mathematical language and notation (MP6). Students continue reasoning abstractly and quantitatively (MP2) as they use their analyses of representations of functions to draw conclusions about the quantities in situations.

## Standards

| | |
|---|---|
| Addressing | **HSA-REI.D.11, HSF-IF.B.4, HSF-IF.B.6** |
| Building Toward | **HSA-REI.D.11** |

## Instructional Routines

- MLR2: Collect and Display
- MLR5: Co-Craft Questions
- MLR8: Discussion Supports

## Student Facing Learning Goals

Let’s compare graphs of functions to learn about the situations they represent.

# 9.1 Population Growth

Warm-up

5 min

## Activity Narrative

In this *Warm-up*, students compare functions by analyzing graphs and statements in function notation. The work here prepares students to make more sophisticated comparisons later in the lesson.

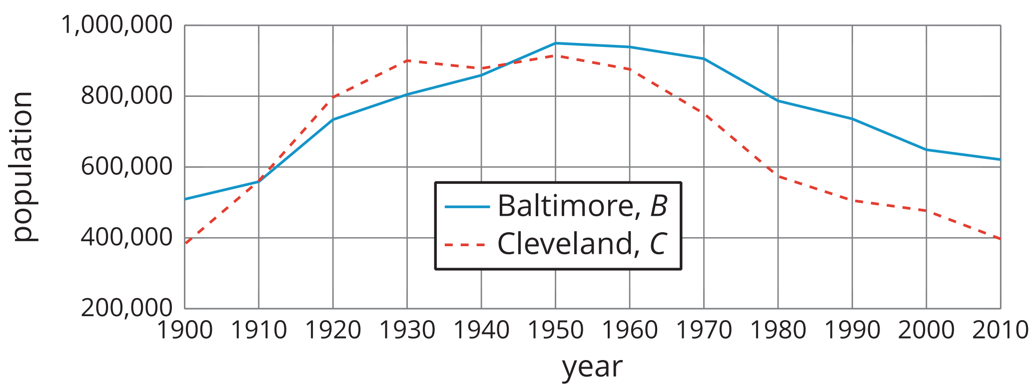
## Standards

Addressing **HSF-IF.B.4**

Building Toward **HSA-REI.D.11**

## Student Task Statement

This graph shows the populations of Baltimore and Cleveland in the 20th century. $B(t)$ is the population of Baltimore in year $t$. $C(t)$ is the population of Cleveland in year $t$.




1. Estimate $B(1930)$, and explain what it means in this situation.
2. Here are pairs of statements about the two populations. In each pair, which statement is true? Be prepared to explain how you know.
    a. $B(2000) > C(2000)$ or $B(2000) < C(2000)$
    b. $B(1900) = C(1900)$ or $B(1900) > C(1900)$
3. Were the two cities' populations ever the same? If so, when?

## Student Response

1. $B(1930)$ is about 800 thousand. It's the population of Baltimore in 1930.
2. a. $B(2000) > C(2000)$
    b. $B(1900) > C(1900)$
3. Yes. The populations were the same in roughly 1910 and 1943.

## Activity Synthesis

Invite students to share their response to the first question. After students give a reasonable estimate of the population of Baltimore (about 800,000), display the statement $B(1930) = 800{,}000$ for all to see. Make sure students can interpret it to mean "In 1930, the population of Baltimore was about 800,000 people."

Next, ask students to explain how they knew which statement in each pair of inequalities is true and how they knew that there were two points in time when Baltimore and Cleveland had the same population.

Ask students how we could use function notation to express that the populations of Baltimore and Cleveland were equal in 1910. If no students mention $B(1910) = C(1910)$ or $B(t) = C(t)$ for $t = 1910$, bring these up and display these statements for all to see.

# 9.2 Wired or Wireless?

20 min

## Activity Narrative

In this activity, students continue to compare two functions by studying graphs and statements in function notation, as well as interpreting them in terms of a situation. They also revisit the meaning of a solution to an equation such as $C(t) = 20$, both abstractly and in context, and apply what they know about average rate of change to compare the trend shown by each graph.

Additionally, the activity draws students' attention to the point where the two graphs intersect, its meaning in context, and its corresponding representation in function notation.

### Standards

Addressing **HSA-REI.D.11, HSF-IF.B.4, HSF-IF.B.6**

### Instructional Routines

- MLR5: Co-Craft Questions

## Launch

Ask students how many of them have a landline phone at home and how many have only cell phones. If students are unfamiliar with landline phones, explain as needed.

Display the graphs for all to see. Discuss questions such as:

- "How would you describe the trends in phone ownership over the years?" (Cell phones are increasing in use, and landlines are decreasing.)
- "How would you describe the shape of the graph of each function?" (Both are roughly linear. The value of $H$ is decreasing over time, so the graph slants downward. The value of $C$ is increasing over time, so the graph slants upward.)
- "In what year did about 60% of homes have a landline?" (2012)

### Access for English Language Learners

*MLR5 Co-Craft Questions.* Keep books or devices closed. Display only the problem stem and graph, without revealing the questions, and ask students to write down possible mathematical questions that could be asked about the situation. Invite students to compare their questions before revealing the task. Ask, "What do these

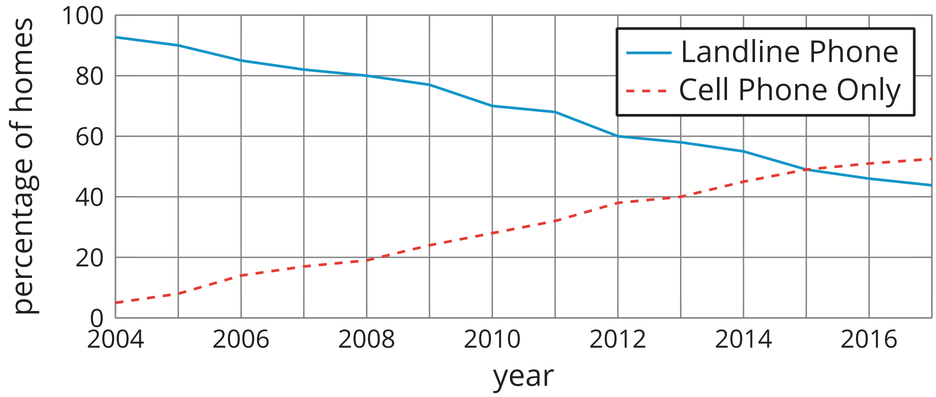
questions have in common? How are they different?" Reveal the intended questions for this task, and invite additional connections.
*Advances: Reading, Writing*

## Access for Students with Disabilities

*Engagement: Develop Effort and Persistence.* Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation, such as "One thing that is the same is . . . ." "One thing that is different is . . . ." "_____ represents _____."
*Supports accessibility for: Language, Social-Emotional Functioning*

## Student Task Statement

$H(t)$ is the percentage of homes in the United States that have a landline phone in year $t$. $C(t)$ is the percentage of homes with *only* a cell phone. Here are the graphs of $H$ and $C$.




1. Estimate $H(2006)$ and $C(2006)$. Explain what each value tells us about the phones.
2. What is the approximate solution to $C(t) = 20$? Explain what the solution means in this situation.
3. Determine if each equation is true. Be prepared to explain how you know.
    a. $C(2011) = H(2011)$
    b. $C(2015) = H(2015)$
4. Between 2004 and 2015, did the percentage of homes with landlines decrease at the same rate at which the percentage of cell-phones-only homes increased? Explain or show your reasoning.

## Student Response

1. $H(2006)$ is about 85, and $C(2006)$ is about 14. In 2006, 85% of homes had a landline phone, and 14% of homes had only a cell phone.
2. Approximately 2008. It means that around 2008, 20% of homes used only cell phones.
3. Sample response:
    a. No, because $C(2011)$ is about 30 and $H(2011)$ is about 70. The percentage of homes with only a landline phone and the percentage of homes with only a cell phone were not equal.
    b. Yes, because in 2015 the percentage of homes with only a landline and the percentage of homes with only a cell phone were equal, as shown by the intersection of the graphs.

4. Sample responses:
    - Yes, one decreased at roughly the same rate as the other one increased.
    - No, the percentage of cell-phones-only homes increased at a rate close to that by which the percentage of landline-owning homes decreased, but not exactly the same.

    Sample reasoning:
    - In 2004, only about 6% of homes had only cell phones. In 2015, about 48% did. The average rate of change for $C$ is $\frac{48-6}{2015-2004} = \frac{42}{11}$, or about 3.8. This means the percentage of homes relying only on cell phones grew by about 3.8% each year.
    - In 2004, about 92% of homes had a landline. In 2015, only about 48% did. The average rate of change for $H$ is $\frac{48-92}{2015-2004} = \frac{-44}{11}$, or -4. This means the percentage of homes that used only landlines fell by 4% each year.

## Building on Student Thinking

Students encounter percentages as the output of a function for the first time in this activity. Some students might think that the output of the functions here must be number of homes, and that they cannot estimate any output values because only percentages are known. Clarify that percent is the unit used in this case, as we are studying how the proportion of the two groups (rather than the actual number of homes in each group) changed over time.

## Are You Ready for More?

1. Explain why the statement $C(t) + H(t) \leq 100$ is true in this situation.
2. What value does $C(t) + H(t)$ appear to take between 2004 and 2017? How much does this value vary in that interval?

## Extension Student Response

1. Each household either has a landline or has no landline and relies only on cell phones, so $H(t)$ and $C(t)$ calculate separate groups of people. As long as there are not rounding errors, $H(t)$ and $C(t)$ together can only be 100 percent, less if there are people with no phone service at all.
2. It appears to take a value a little less than 100, maybe close to 97. For example, the two graphs meet below the midway point (50). At the beginning and the end of the shown interval, they add up to a little less than 100 as well. The sum is not constant (the shapes of the two graphs are different) but does not vary much, so the percentage of people with no phone service is small for the entire period.

## Activity Synthesis

Focus the discussion on the meaning of equations such as $C(t) = 20$ and $C(2015) = H(2015)$, and on the meaning of the average rate of change of each function.

Select students to share their responses. Highlight the following points, if not already mentioned in students' explanations:

- $C(t) = 20$ means "in year $t$, 20% of homes relied only on cell phones," and based on the graph of $C$, the value of $t$ that makes that statement true is 2008.
- $C(2015) = H(2015)$ means "in year 2015, the percentage of homes with only cell phones and the percentage of homes with only landlines are equal." We know this is true because at $t = 2015$ the two graphs intersect, which also

means they share the same output value.

- The average rate change for $C$ is positive because in the measured interval, the value of $C$ increased overall. The average rate of $H$ is negative because the value of $H$ decreased overall.
- An average rate of change of 3.8% per year for $C$ means that the percentage of homes relying only on cell phones grew by about 3.8% each year.
- An average rate of change of -4% per year means the percentage of homes that used only landlines fell by 4% each year.

If time permits, discuss with students:

- "The average rate of change for $C$ is 4% per year, while the average rate of change for $H$ is -3.8% per year, which is very close to -4%. Why might the rate at which one increased be so close to the rate at which the other decreased? Could it be a coincidence?" (One possible explanation is that as people relied more on cell phones, they relied less on landlines, and they discontinued using landlines around the same time they acquired new cell phones. These people essentially went from one group to the other.)

# 9.3 Audience of TV Shows

15 min

Optional

## Activity Narrative

Previously, students compared functions by analyzing their graphs on the same coordinate plane. Each graph was a continuous graph.

In this optional activity, students compare functions represented in separate graphs. Each graph is a discrete graph, showing the viewership of three TV shows as functions of the episode number. Students interpret features of the graphs and relate them to descriptions about the shows and to statements in function notation. They use their analyses to draw conclusions about the popularity of the shows and to sketch a possible graph for a fourth TV show.

The work here requires students to make sense of quantities and their relationships while attending to their representations (MP2). In sketching a graph that matches a description, students need to be careful about showing correspondence to the quantities in the situation (MP6), including by using appropriate scale and marks.

## Standards

Addressing **HSF-IF.B.4**

## Instructional Routines

- MLR8: Discussion Supports

## Launch

Arrange students in groups of 2. Give students a few minutes of quiet work time, then time to discuss their thinking with their partner. Follow with a whole-class discussion.

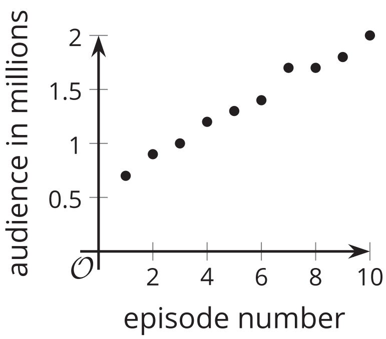
## Access for English Language Learners

*MLR8 Discussion Supports.* Students should take turns finding a match and explaining their reasoning to their partner. Display the following sentence frames for all to see: "I noticed _____, so I matched . . . ." Encourage students to challenge each other when they disagree.
*Advances: Speaking, Representing*

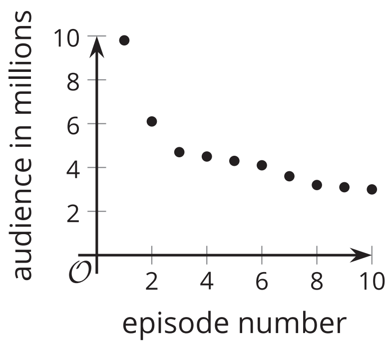
## Student Task Statement

The number of people who watched a TV episode is a function of that show's episode number. Here are three graphs of three functions—$A$, $B$, and $C$—representing three different TV shows.

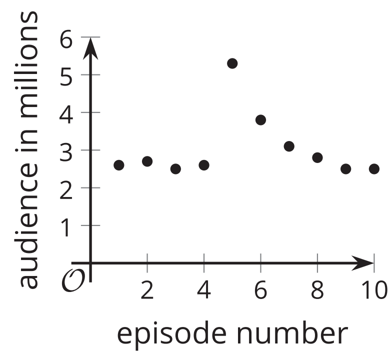
**Show A**



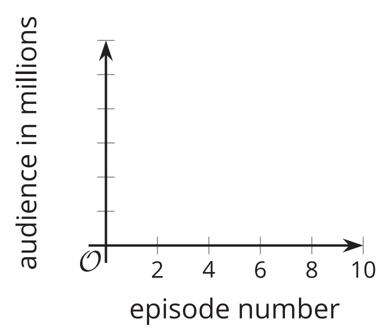

**Show B**




**Show C**

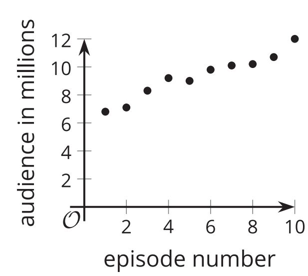



1. Match each description with a graph that could represent the situation described. One of the descriptions has no corresponding graph.
   a. This show has a good core audience. They had a guest star in the fifth episode that brought in some new viewers, but most of them stopped watching after that.
   b. This show is one of the most popular shows, and its audience keeps increasing.
   c. This show has a small audience, but the show is improving, so more people are noticing.
   d. This show started out huge. Even though it feels like it crashed, it still has more viewers than another show.
2. Which is greatest, $A(7)$, $B(7)$, or $C(7)$? Explain what the answer tells us about the shows.
3. Sketch a graph of the viewership of the fourth TV show that did not have a matching graph.




## Student Response

1. a. graph of function $C$
   b. no match
   c. graph of function $A$
   d. graph of function $B$
2. $B(7)$. Show B has the most viewed 7th episode of these 3 shows.
3. See sample graph.



## Activity Synthesis

Focus the discussion on how students made their matches. Ask students to explain how parts of the descriptions and features of the graphs led them to believe that a pair of representations belong together.

Next, invite students to share their graph of the fourth TV show. Display the graphs for all to see, and discuss how the graphs are alike and how they are different. Because each graph is created using the same description, they should share some common features. If they look drastically different, solicit possible reasons. (Possible explanations include differences in interpretation of the description or in the choice of scale for the vertical axis, errors in reading the description, and plotting errors.)

If time permits, ask students,

- "How are the graphs or the functions in this activity different from those in earlier activities?" (They show points rather than lines. The input is episode number, while in other cases, the input is time. Each function is shown on a different coordinate plane.)
- "How is the work of comparing functions here like comparing functions in earlier activities?" (They all involve comparing the output values of points on a graph, interpreting the points, making sense of verbal descriptions, and some estimating.)
- "How is the work of comparing functions here different from in earlier activities?" (Here we are comparing function values across three separate graphs, which is trickier than when the graphs are all on the same coordinate plane. Figuring out which function has a greater or lesser value was also easier when the functions were represented with lines or curves. It is harder to do with a set of points, especially when they are not in the same image.)

# 9.4 Functions $f$ and $g$

10 min

## Activity Narrative

In this activity, students compare graphs and statements that represent functions without a context. Because no concrete information is given, students need to rely on their understanding of function notation and points on a graph to make comparisons and to interpret intersections of the graphs.

Previously, students saw equations such as $f(8) = g(8)$ and interpreted them in terms of a situation. Here, they begin to reason more abstractly about statements of the form $f(x) = g(x)$ and relate them to one or more points where the graphs of $f$ and $g$ intersect. They see that a value of $x$ that makes this equation true, or a solution to the equation, is the input value of such an intersection.

### Standards

Addressing **HSA-REI.D.11, HSF-IF.B.4**

### Instructional Routines

- MLR2: Collect and Display

## Launch

### Access for English Language Learners

*MLR2 Collect and Display.* Collect the language students use to share their interpretations of equations written as $f(x) = g(x)$. Display words and phrases, such as "the values are equal," "intersect," "horizontal value," "input

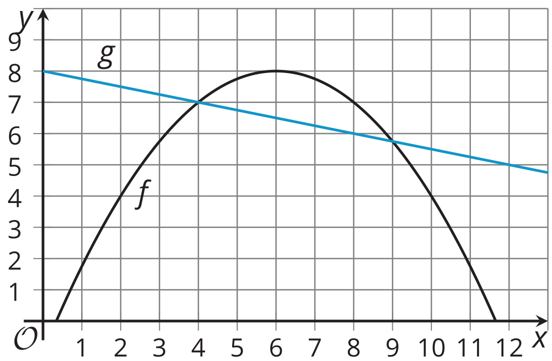
value," and "common point." During the *Activity Synthesis*, invite students to suggest ways to update the display: "What are some other words or phrases we should include?" Invite students to borrow language from the display as needed.
*Advances: Conversing, Reading*

## Student Task Statement

1. Here are graphs that represent two functions, $f$ and $g$.

   Decide which function value is greater for each given input. Be prepared to explain your reasoning.

   a. $f(2)$ or $g(2)$
   b. $f(4)$ or $g(4)$
   c. $f(6)$ or $g(6)$
   d. $f(8)$ or $g(8)$




2. Is there a value of $x$ at which the equation $f(x) = g(x)$ is true? Explain your reasoning.
3. Identify at least two values of $x$ at which the inequality $f(x) < g(x)$ is true.

## Student Response

1. a. $g(2)$ is greater.
   b. They appear to be equal.
   c. $f(6)$ is greater.
   d. $f(8)$ is greater.
2. Yes. Sample reasoning: If the graphs intersect when $x$ is 4 and when it is 9, then $f(4) = g(4)$ and $f(9) = g(9)$.
3. Sample response: 1, 3, 10, 11, or other $x$-values that are less than 4 or greater than 9.

## Activity Synthesis

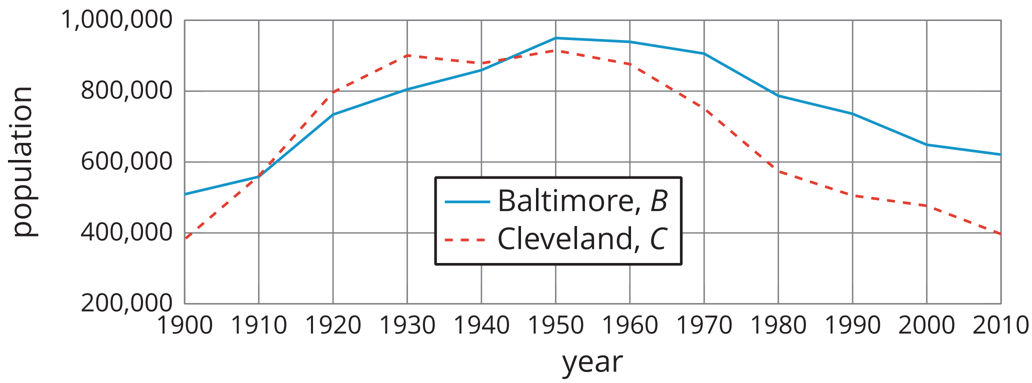
Display the graphs of $f$ and $g$ for all to see. Invite students to share their responses to the first set of questions. As they point out the greater function value in each pair, mark the point on the graph and write a corresponding statement in function notation: $f(2) < g(2)$, $f(4) = g(4)$, $f(6) > g(6)$, and $f(8) > g(8)$. Emphasize that the function value that is greater in each pair has a higher vertical value on the coordinate plane for the same input value.

Next, discuss if there are values of $x$ that make $f(x) = g(x)$ true and how we can tell. Point out that earlier we wrote $f(4) = g(4)$ because the values of $f$ and $g$ are equal when the input is 4. This means that $(4, f(4))$ and $(4, g(4))$ are coordinates of the same point.

So we can interpret an equation such as $f(x) = g(x)$ to mean that the values of $f$ and $g$ are equal when the input is $x$, and that $x$ must be the horizontal value of the intersection of both graphs, which is a point that they share.

## Lesson Synthesis

Refer back to the population graphs for Baltimore and Cleveland from 1900 to 2010, which students saw in the *Warm-up*. Display the graphs for all to see.




Present students with the following statements, one at a time, about the populations of the two cities. Tell students that their job is to explain how they could tell from the graphs that each statement is true and to translate each verbal description into a statement with the same meaning but written in function notation.

Consider using a three-column graphic organizer (as shown here and in the *Lesson Summary*) to organize students' responses.

| What can we tell about the populations? | How can we tell? | How can we convey this with function notation? |
|---|---|---|
| In 2010, Baltimore had more people than Cleveland. | | |
| Baltimore and Cleveland had the same population twice in the past century, in 1910 and around 1944. | | |
| After the mid-1940s, Cleveland has had a smaller population than Baltimore. | | |
| In the first half of the 20th century, the population of Cleveland grew at a faster rate than that of Baltimore. | | |
| Since 1950, the population of Cleveland has dropped at a faster rate than that of Baltimore. | | |

## 9.5 A Toy Rocket and a Drone Again

Cool-down

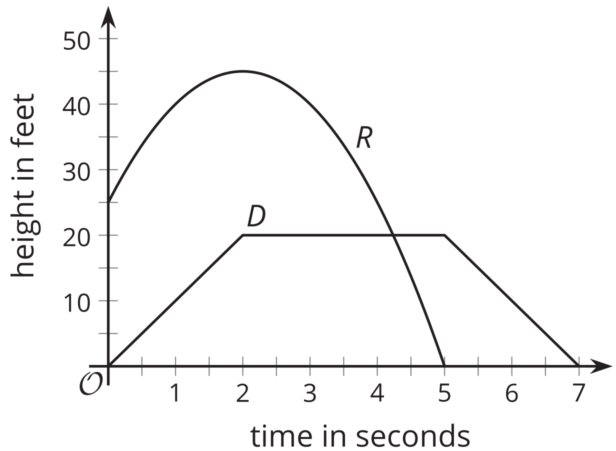
5 min

### Standards

Addressing **HSA-REI.D.11, HSF-IF.B.4**

## Student Task Statement

Functions $R$ and $D$ give the height, in feet, of a toy rocket and a drone, $t$ seconds after they are released. Here are the graphs of $R$ (for the rocket) and $D$ (for the drone).




1. Which of the inequalities is true: $R(2) > D(2)$ or $R(2) < D(2)$?
2. What was the height of the drone when the toy rocket hit the ground?
3. For what value of $t$ is $R(t) = D(t)$ true? What does this tell you about the drone and the toy rocket?

## Student Response

1. $R(2) > D(2)$
2. 20 feet
3. About $t = 4.25$. After about 4.25 seconds, the toy rocket and the drone are at the same height.

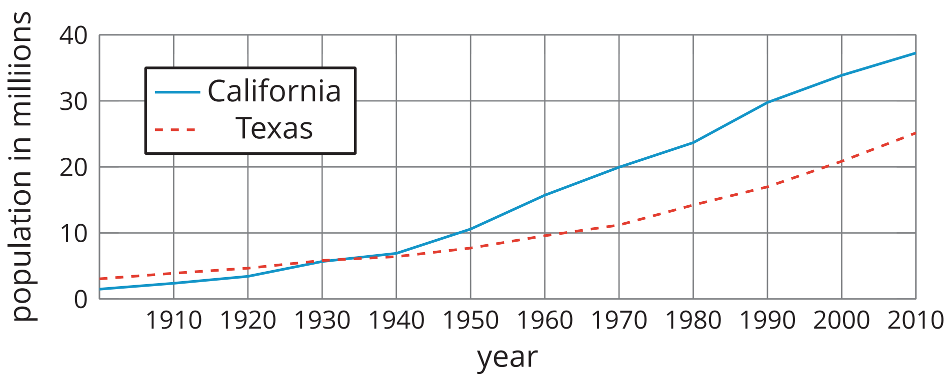
## Responding to Student Thinking

Points to Emphasize
If most students struggle with comparing graphs using function notation, as opportunities arise, revisit the meaning of function notation and how it is used to represent graphs. For example, this concept can be reviewed while discussing the following practice problem:

Algebra 1, Unit 5, Lesson 9, Practice Problem 2

## Lesson 9 Summary

Graphs are very useful for comparing two or more functions. Here are graphs of functions $C$ and $T$, which give the populations (in millions) of California and Texas in year $x$.

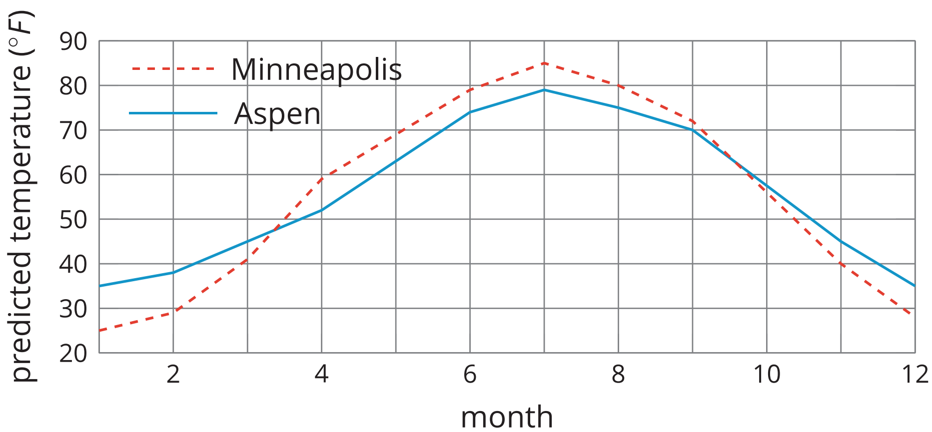



| What can we tell about the populations? | How can we tell? | How can we convey this with function notation? |
| --- | --- | --- |
| In the early 1900s, California had a smaller population than Texas. | The graph of $C$ is below the graph of $T$ when $x$ is 1900. | $C(1900) < T(1900)$ |
| Around 1935, the two states had the same population of about 5 million people. | The graphs intersect at about $(1935, 5)$. | $C(1935) = 5$ and $T(1935) = 5$, and $C(1935) = T(1935)$. |
| After 1935, California has had more people than Texas. | When $x$ is greater than 1935, the graph of $C(x)$ is above that of $T(x)$. | $C(x) > T(x)$ for $x > 1935$ |
| Both populations have increased over time, with no periods of decline. | Both graphs slant upward from left to right. | |
| From 1900 to 2010, the population of California has risen faster than that of Texas. California had a greater average rate of change. | If we draw a line to connect the points for 1900 and 2010 on each graph, the line for $C$ has a greater slope than that for $T$. | $\frac{C(2010) - C(1900)}{2010 - 1900} > \frac{T(2010) - T(1900)}{2010 - 1900}$ |

# Lesson 9 Practice Problems

## 1 Student Task Statement

$A(t)$ is the average high temperature in Aspen, Colorado, $t$ months after the start of the year. $M(t)$ is the average high temperature in Minneapolis, Minnesota, $t$ months after the start of the year. Temperature is measured in degrees Fahrenheit.




Which function had the higher average rate of change between the beginning of January and middle of March? What does this mean about the temperature in the two cities?

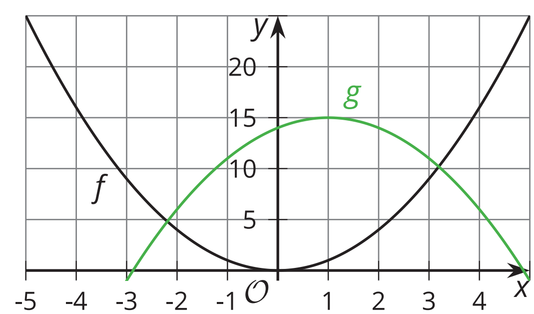
### Solution

Minneapolis had the higher rate of change. This means that between January and March the average high temperature increased faster in Minneapolis than in Aspen.

## 2 Student Task Statement

Here are two graphs representing functions $f$ and $g$.

Select **all** statements that are true about functions $f$ and $g$.




A. $f(0) > g(0)$

B. There are two values of $x$ where $f(x) = g(x)$.

C. $f(-1) < g(-1)$

D. $f(-3) > g(4)$

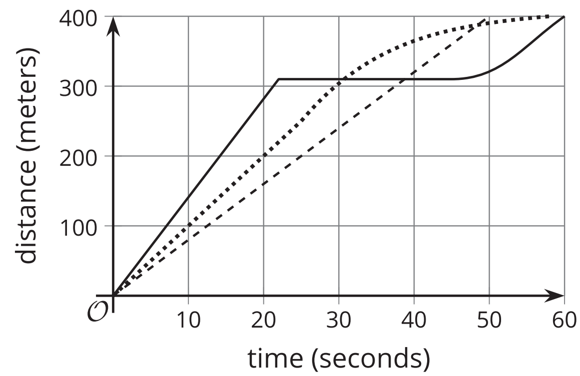
## Solution

B, C, D

## 3 Student Task Statement

The three graphs represent the progress of three runners in a 400-meter race.

The solid line represents runner A. The dotted line represents runner B. The dashed line represents runner C.




a. One runner ran at a constant rate throughout the race. Which one?
b. A second runner stopped running for a while. Which one? During which interval of time did that happen?
c. Describe the third runner's race. Be as specific as possible.
d. Who won the race? Explain how you know.

## Solution

a. Runner C
b. Runner A. It happened from approximately 22 seconds to about 45 seconds after the race began.
c. Sample response: Runner B ran at a greater constant rate than runner C for the first 27 or 28 seconds but then began to slow down and to trail behind runner C. This runner finished second.
d. Runner C. Sample reasoning: The runner reached 400 meters in 50 seconds. The other two runners reached 400 meters in about 58 seconds and 60 seconds.

## 4 from Unit 5, Lesson 4

## Student Task Statement

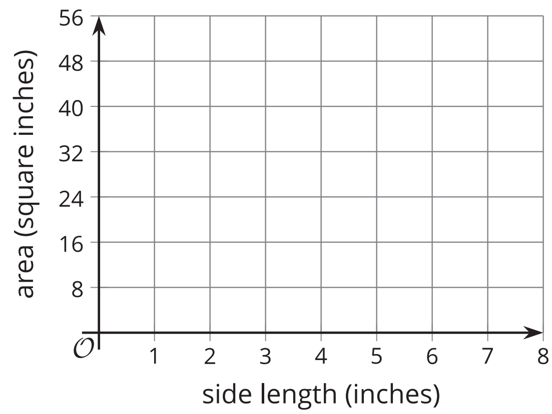
Function $A$ gives the area, in square inches, of a square with side length $x$ inches.

a. Complete the table.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $A(x)$ | | | | | | | |

b. Represent function $A$ using an equation.

c. Sketch a graph of function $A$.



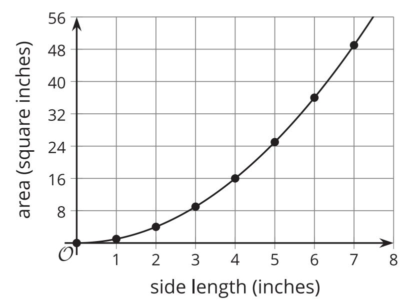

### Solution

a.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $A(x)$ | 0 | 1 | 4 | 9 | 16 | 25 | 36 |

b. $A(x) = x^2$

c. See graph.




**5** from Unit 5, Lesson 5

### Student Task Statement

Function $f$ is represented by $f(x) = 5(x + 11)$.

a. Find $f(\text{-}2)$.

b. Find the value of $x$ such that $f(x) = 90$ is true.

### Solution

a. $f(\text{-}2) = 5(\text{-}2 + 11) = 5(9)$, so $f(\text{-}2)$ is 45.

b. $5(x + 11) = 90$, so $x + 11 = 18$, and $x = 7$.