Unit 5, Lesson 22

Volume as a Function of . . .

Goals

- Describe (orally) how a change in the radius of a sphere affects the volume.
- Interpret (orally and in writing) functions that represent the volumes of a spheres, cones, and cylinders, using different representations.

Learning Targets

- I can compare functions about volume represented in different ways.

Lesson Narrative

The purpose of this optional culminating lesson is to give students more experience working with nonlinear functions that arise out of the work students have been doing with the volumes of cylinders, cones, and spheres.

The *Warm-up* invites students to contrast how knowing the height of a cylinder does not fix the volume or radius of the cylinder in the same way it does for a sphere of fixed height. Next, students continue to work with spheres and reason about how scaling the radius affects the volume of a sphere. First they repeat calculations for the volume of spheres of various radii, and then they generalize for spheres with radius r (MP8).

In the last activity, students work with three different functions (represented three different ways) showing how the height of water in three different shapes is a function of the volume of water being poured into the shape. They consider questions such as:

- Which container is largest?
- At what volume of water poured is the height of water in the containers the same?
- For what range of water volume poured does a particular container have the greatest height of water?
- How do the representations show the maximum height of each container?

Standards

Addressing **8.F.A, 8.G.C.9**

Instructional Routines

- MLR1: Stronger and Clearer Each Time
- MLR8: Discussion Supports
- Notice and Wonder

Required Materials

Materials to Gather

- Straightedges: Activity 3

Required Preparation

Lesson:

Provide access to straightedges for the activity "A Cylinder, a Cone, and a Sphere."

Student Facing Learning Goals

Let's compare water heights in different containers.

22.1 Missing Information?

Warm-up

5 min

Activity Narrative

In this *Warm-up*, students reason about the volume and dimensions of a cylinder based on information given about a sphere of the same height. The purpose of this *Warm-up* is for students to recognize when they do not have enough information to reach a single answer. While they can determine the height of the cylinder, the radius is unknown, which means the volume of the cylinder could be anything.

Standards

Addressing **8.G.C.9**

Launch

Give students quiet work time, and follow with a whole-class discussion.

Student Task Statement

A cylinder and sphere have the same height.

1. If the sphere has a volume of 36π cubic units, what is the height of the cylinder?
2. What is a possible volume for the cylinder? Be prepared to explain your reasoning.

Student Response

1. 6 units
2. Students could answer anything here, as the radius is undetermined. Knowing the height, though, creates a function with variables V and r. Students just need to pick an r, and they have a volume that works. Going the other way is harder but possible, and working with square roots is left for a future unit.

Building on Student Thinking

Some students may not understand why the second problem talks about "possible" volumes if they assume the radius of the cylinder is the same as the sphere. Ask these students to explain how they found the radius of the cylinder so that they notice that the problem never gives that information.

Activity Synthesis

The purpose of this discussion is to make sure students understand that the volume of the cylinder could be anything. Ask students to share how they calculated the height of the cylinder. If any students made a sketch, display these for all to see.

Select at least 5 students to give a possible volume for the cylinder, and record these for all to see. Ask students,

- "Why do we not know what the volume of the cylinder is?" (We don't know the radius, only the height, so the volume could be anything.)
- "Is knowing the height of a sphere enough information to determine the volume?" (Yes. The volume of a sphere is based on the radius, which is half the height.)

Tell students that in the next activity, they will investigate how changes to the radius of a sphere changes the volume of the sphere.

22.2 Scaling Volume of a Sphere

15 min

Activity Narrative

Building on work in previous lessons in which students investigated how changing dimensions affects the volume of a shape, in this activity, students scale the radius of a sphere and compare the resulting volumes. They use repeated reasoning from calculations in a table to predict how doubling or halving the radius of a sphere affects the volume (MP8).

Monitor for students using different reasoning to answer the last question. For example, to get the volume of the smaller sphere, some students may calculate the radius of the larger sphere in order to find $\frac{1}{5}$ of that value, while others may reason about the volume formula and how volume changes when the radius goes from r to $\frac{1}{5}r$.

Standards

Addressing **8.G.C.9**

Instructional Routines

- MLR1: Stronger and Clearer Each Time

Launch

Arrange students in groups of 2. Give 2–3 minutes of quiet work time to complete the first problem on their own, then time to discuss their solutions with a partner. Groups then finish the remaining problems together and follow with a whole-group discussion.

Access for Students with Disabilities

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "It looks like . . . ." "Is it always true that . . .?" and "We can agree that . . . ."
Supports accessibility for: Language, Organization

Access for English Language Learners

MLR1 Stronger and Clearer Each Time. Before the whole-class discussion, give students time to meet with 2–3 partners to share and get feedback on their first draft response to "What happens to the volume of this sphere if its radius is halved?" Invite listeners to ask questions and give feedback that will help their partner clarify and strengthen their ideas and writing. Give students 3–5 minutes to revise their first draft based on the feedback they receive.
Advances: Writing, Speaking, Listening

Student Task Statement

1. Fill in the missing volumes in terms of π. Add two more radius and volume pairs of your choosing.

radius (cm)	1	2	3	$\frac{1}{2}$	$\frac{1}{3}$	100			r
volume (cm^3)	$\frac{4}{3}\pi$								

   a. How does the volume of a sphere with radius 2 cm compare to the volume of a sphere with radius 1 cm?

   b. How does the volume of a sphere with radius $\frac{1}{2}$ cm compare to the volume of a sphere with radius 1 cm?

2. A sphere has a radius of length r.

   a. What happens to the volume of this sphere if its radius is doubled?

   b. What happens to the volume of this sphere if its radius is halved?

3. Sphere Q has a volume of 500 cm^3. Sphere S has a radius $\frac{1}{5}$ as large as Sphere Q. What is the volume of Sphere S?

Student Response

1. Answers vary for the radius and volume pairs of students' choosing. The rest of the table should look as follows:

radius	1	2	3	$\frac{1}{2}$	$\frac{1}{3}$	100			r
volume	$\frac{4}{3}\pi$	$\frac{32}{3}\pi$	36π	$\frac{1}{6}\pi$	$\frac{4}{81}\pi$	$\frac{4{,}000{,}000}{3}\pi$			$\frac{4}{3}\pi r^3$

   a. The volume of a sphere with radius 2 is 8 times as large as the volume of a sphere with radius 1.

   b. The volume of a sphere with radius $\frac{1}{2}$ is $\frac{1}{8}$ as large as the volume of a sphere with radius 1.

2. a. A sphere with two times the radius will have volume $V = \frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$, which is 8 times larger than the original.

   b. A sphere with a radius half the original will have volume $V = \frac{4}{3}\pi\left(\frac{r}{2}\right)^3 = \frac{1}{6}\pi r^3$, which is $\frac{1}{8}$ of the original volume.

3. 4 cm^3. If the radius is $\frac{1}{5}$ as large, the volume will be $\left(\frac{1}{5}\right)^3$ as large, or $\frac{1}{125}$. Therefore, the volume of Sphere S will be

$500\left(\frac{1}{125}\right)$, or 4, cm^3.

Building on Student Thinking

If you notice students interpreting $(2r)^3$ as $2 \cdot r^3$ in the second problem, consider asking:

- "Can you explain how you calculated that value for the volume of a sphere with double the radius?"
- "How does your result match with the values you calculated in the table?"

Activity Synthesis

The purpose of this discussion is for students to understand that since the value of r is cubed in the formula for volume, changing the radius of a sphere affects the volume by the cubed value of that change.

Display the completed table for all to see, and invite groups to share a pair they added to the table along with their responses to the first two problems.

Select previously identified students to share their responses to the last question. If students did not use the formula for volume of a sphere to reason about the question (that is, by reasoning that if the radius is $\frac{1}{5}$ as large, then $v = \frac{4}{3}\pi\left(\frac{1}{5}r\right)^3 = \frac{1}{125}\left(\frac{4}{3}\pi r^3\right)$), ask students to consider their responses to the second part of the second question, but replace $\frac{1}{2}$ with $\frac{1}{5}$.

22.3 A Cylinder, a Cone, and a Sphere

25 min

Activity Narrative

The purpose of this activity is for students to bring together several ideas they have been working with, in particular, calculating volume and dimensions of round objects, comparing functions represented in different ways, interpreting the slope of a graph in context, reasoning about specific function values, and reasoning about when functions have the same value.

While students are only instructed to add a graph of the cylinder to the given axes, monitor for students who also plot the values for the sphere to share their graphs during the *Activity Synthesis*.

Standards

Addressing **8.F.A, 8.G.C.9**

Instructional Routines

- MLR8: Discussion Supports
- Notice and Wonder

Launch

Tell students to close their books or devices (or to keep them closed). Display the graph from the *Task Statement*. Tell students that the graph represents water filling a container. Give students 1 minute of quiet think time, and ask them to be prepared to share at least one thing they notice and one thing they wonder. Record and display responses without editing or commentary for all to see. If possible, record the relevant reasoning on or near the graph. Possible responses:

Students may notice:

- The graph is not a linear function.
- The graph is a piecewise function.
- The maximum height of the function is 6 inches.

Students may wonder:

- What is the shape of the container?
- What is happening when the height stops at 6 inches?
- Why does the volume increase but the height stays constant on the far right?

Discuss possible responses for questions that students wondered. If what would happen if someone kept pouring water into a container even though the water level had reached the top does not come up during the conversation, ask students to discuss this idea. Ensure students understand that the height stops at 6 inches because that is how tall the container is. Any more water poured into the container at that point just overflows.

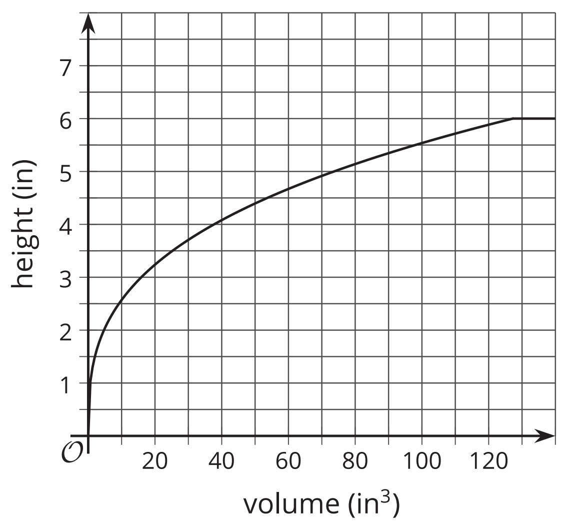
Arrange students in groups of 2–3, and provide access to straightedges. Tell students to open their books or devices, then give groups work time, and follow with a whole-class discussion.

Student Task Statement

Three containers of the same height were filled with water at the same rate. One container is a cylinder, one is a cone, and one is a sphere.

As they were filled, the relationship between the volume of water and the height of the water was recorded in different ways, shown here:

- Cylinder: $h = \frac{V}{4\pi}$
- Cone:



- Sphere:

volume (in^3)	height (in)
0	0
8.38	1
29.32	2
56.55	3
83.76	4
104.72	5
113.04	6
120	6
200	6

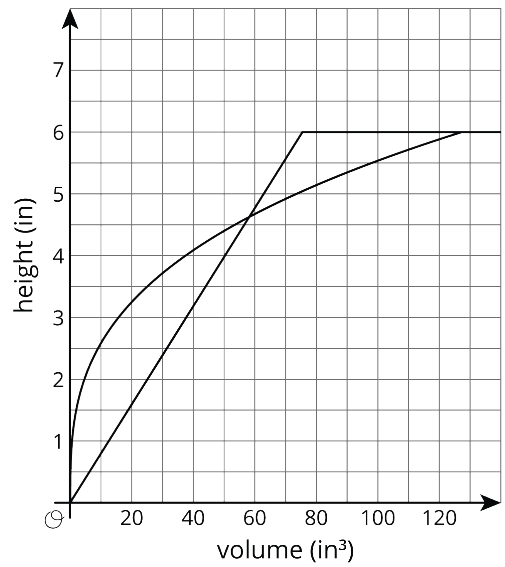
1. The maximum volume of water the cylinder can hold is 24π. What is the radius of the cylinder?
2. Graph the relationship between the volume of water poured into the cylinder and the height of water in the cylinder on the same axes as the cone. What does the

slope of this line represent?

3. Which container can fit the largest volume of water? The smallest?
4. About how much water does it take for the cylinder and the sphere to have the same height? The cylinder and the cone? Explain how you know.
5. For what approximate range of volumes is the height of the water in the cylinder greater than the height of the water in the cone? Explain how you know.
6. For what approximate range of volumes is the height of the water in the sphere less than the height of the water in the cylinder? Explain how you know.

Student Response

1. 2 in. We know the maximum volume of water is 24π in^3, and the height of the cylinder is 6 in since it is the same as the sphere. As such, $r = 2$ since $24\pi = \pi r^2 \cdot 6$ must be true.
2.



The slope of the line is $\frac{1}{4\pi}$. This means that for each volume increase of 1 in^3, the height increases by $\frac{1}{4\pi}$ in.

3. The cone can hold the most water since the graph shows a volume of approximately 127 in^3 when the cone is full. The cylinder can hold the smallest amount of water, 24π in^3.
4. Since the data for volume and height of the spherical container is discrete, we can only say that the volumes and heights must be equal somewhere between the two points at $(8.38, 1)$ and $(29.32, 2)$. An approximation between these two points is that when the volume is about 20 in^3, the height is about 1.5 in for both the cylinder and sphere. For the cylinder and cone, we can see the intersection of the two points on the graph from the second part, which occurs approximately when the volume is 58 in^3 and the height is 4.5 in.
5. When the volume is between 58 in^3 and 125 in^3, the height of water in the cylinder is greater than the height of water in the cone.
6. When the volume is between about 20 in^3 (or the data point chosen for the first part of the fourth part) and 113 in^3, the height of water in the cylinder is greater than the height of water in the sphere.

Building on Student Thinking

If students are not sure how to compare some of the different representations, consider asking:

- "Tell me more about what the values in the table tell you."
- "How could you use a graph to think of the relationship between the volume of water and the height of the water for the sphere in a different way?"

Activity Synthesis

The purpose of this discussion is for students to share how they compared the different representations or made a new representation to answer the questions. Begin by surveying students for their answers to the third part about which container can fit the largest volume of water and which container can fit the smallest. Record the results for all to see. Invite 2–3 to share their reasoning.

Next, select previously identified students who graphed the data from the table in order to complete the problems. Display 1–2 of these representations for all to see.

Here are some questions to further student thinking about the graphs of the three shapes:

- "If I showed you this graph without telling you which function represented each shape, how could you figure out which one represents the cone, the cylinder, and the sphere?" (The graph of the cylinder must be linear since the shape of the container never changes as the water level rises. The graph of the cone would first fill quickly but then fill more slowly as it got wider near the top. (Note: It helps to know the cone is tip down here.) The graph of the sphere would change partway through since it would start fast, slow down toward the middle where the sphere is widest, then speed up again as the sphere narrows, and the table data, or discrete points, is the only representation that does that.)
- "Can you use the information provided about each function to determine the radius of the sphere and cone? Use the fact that the actual volume of the cone is 127.23 in^3." (The radius of the sphere is half the height, so the radius is 3 in. The cone has a volume of 127.23 in^3, so we can tell that r^2 is about 20.25. We can use guess and check to find r since 20.25 is between 16 and 25. We can check what happens when $r = 4.5$, which gives a value of 20.25 for r^2. So the radius of the cone is about 4.5 in.)

Access for English Language Learners

MLR8 Discussion Supports. At the appropriate time, give students 2–3 minutes to make sure that everyone in their group can explain how to answer the question "If I showed you this graph without telling you which function represented each shape, how could you figure out which one represents the cone, the cylinder, and the sphere?" Invite groups to rehearse what they will say when they share with the whole class.
Advances: Speaking, Conversing, Representing