Deriving the Quadratic Formula

Let's find out where the quadratic formula comes from.

19.1 Studying Structure

Here are some perfect squares in factored and standard forms, and expressions showing how the two forms are related.

1. Complete the table.

factored form		standard form
$(3x-4)^2$	$(3x)^2+2(\_\_\_x)(\_\_\_)+(\_\_\_)^2$	$9x^2-24x+16$
$(5x+\_\_\_\_)^2$	$(\_\_\_x)^2+2(\_\_\_x)(\_\_\_)+(\_\_\_)^2$	$25x^2+30x+\_\_\_\_$
$(kx+m)^2$	$(\_\_\_x)^2+2(\_\_\_x)(\_\_\_)+(\_\_\_)^2$	$\_\_\_x^2+\_\_\_x+\_\_\_$

2. Look at the expression in the last row of the table. If ax^2+bx+c is equivalent to $(kx+m)^2$, how are a, b, and c related to k and m?

19.2 Complete the Square Using a Placeholder

1. One way to solve the quadratic equation $x^2 + 5x + 3 = 0$ is by completing the square. A partially solved equation is shown here. Study the steps.

   Then, knowing that P is a placeholder for $2x$, continue to solve for x without evaluating any part of the expression. Be prepared to explain each step.

$x^2 + 5x + 3 = 0$	original equation
$4x^2 + 20x + 12 = 0$	Multiply each side by 4.
$4x^2 + 20x = -12$	Subtract 12 from each side.
$(2x)^2 + 10(2x) = -12$	Rewrite $4x^2$ as $(2x)^2$ and $20x$ as $10(2x)$.
$P^2 + 10P = -12$	Use P as a placeholder for $2x$.

$$P^2 + 10P + \_\_\_\_^2 = -12 + \_\_\_\_^2$$

$$(P + \_\_\_\_)^2 = -12 + \_\_\_\_^2$$

$$P + \_\_\_\_ = \pm\sqrt{-12 + \_\_\_\_^2}$$

$$P = \_\_\_\_ \pm \sqrt{-12 + \_\_\_\_^2}$$

$$P = \_\_\_\_ \pm \sqrt{\_\_\_\_^2 - 12}$$

$$2x = \_\_\_\_ \pm \sqrt{\_\_\_\_^2 - 12}$$

$$x =$$

2. Explain how the solution is related to the quadratic formula.

19.3 Decoding the Quadratic Formula

Here is one way to make sense of how the quadratic formula came about. Study the derivation until you can explain what happens in each step. Record your explanation next to each step.

$$ax^2 + bx + c = 0$$

$$4a^2x^2 + 4abx + 4ac = 0$$

$$4a^2x^2 + 4abx = \text{-}4ac$$

$$(2ax)^2 + 2b(2ax) = \text{-}4ac$$

$$M^2 + 2bM = \text{-}4ac$$

$$M^2 + 2bM + b^2 = \text{-}4ac + b^2$$

$$(M + b)^2 = b^2 - 4ac$$

$$M + b = \pm\sqrt{b^2 - 4ac}$$

$$M = \text{-}b \pm \sqrt{b^2 - 4ac}$$

$$2ax = \text{-}b \pm \sqrt{b^2 - 4ac}$$

$$x = \frac{\text{-}b \pm \sqrt{b^2 - 4ac}}{2a}$$

Are you ready for more?

Here is another way to derive the quadratic formula by completing the square.

- First, divide each side of the equation $ax^2 + bx + c = 0$ by a to get $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.
- Then, complete the square for $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

1. The beginning steps of this approach are shown here. Briefly explain what happens in each step.

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	original equation
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	[1]
$x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$	[2]
$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$	[3]
$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$	[4]
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	[5]
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	[6]
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$	[7]

2. Continue the solving process until you have the equation $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Lesson 19 Summary

Recall that any quadratic equation can be solved by completing the square. The quadratic formula is essentially what we get when we put all the steps taken to complete the square for $ax^2 + bx + c = 0$ into a single expression.

When we expand a squared factor like $(3x + 5)^2$, the result is $(3x)^2 + 2(5)(3x) + 25$. Notice how $(3x)$ appears in two places. If we replace $(3x)$ with another letter, like P, we have $P^2 + 10P + 25$, which is a recognizable perfect square.

Likewise, if we expand $(kx + m)^2$, we have $(kx)^2 + 2m(kx) + m^2$. Replacing kx with P gives $P^2 + 2mP + m^2$, also a recognizable perfect square.

To complete the square is essentially to make one side of the equation have the same structure as $(kx)^2 + 2m(kx) + m^2$. Substituting a letter for (kx) makes it easier to see what is needed to complete the square. Let's complete the square for $ax^2 + bx + c = 0$!

- Start by subtracting c from each side.

$$ax^2 + bx = -c$$

- Next, let's multiply both sides by $4a$, which is allowed because $a \neq 0$. On the left, this gives $4a^2$, a perfect square for the coefficient of x^2.

$$4a^2x^2 + 4abx = -4ac$$

- $4a^2x^2$ can be written $(2ax)^2$, and $4abx$ can be written $2b(2ax)$.

$$(2ax)^2 + 2b(2ax) = -4ac$$

- Let's replace $(2ax)$ with the letter P.

$$P^2 + 2bP = -4ac$$

- b^2 is the constant term that completes the square, so let's add b^2 to each side.

$$P^2 + 2bP + b^2 = -4ac + b^2$$

- The left side is now a perfect square and can be written as a squared factor.

$$(P + b)^2 = b^2 - 4ac$$

- The square roots of the expression on the right are the values of $P + b$.

$$P + b = \pm\sqrt{b^2 - 4ac}$$

  Once P is isolated, we can write $2ax$ in its place and solve for x.

$$P = -b \pm \sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

- The solution is the quadratic formula!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$