Unit 4, Lesson 4

Comparing Quadratic and Exponential Functions

Goals

- Use graphs, tables, and calculations to show that exponential functions eventually overtake quadratic functions.

Learning Targets

- I can explain using graphs, tables, or calculations that exponential functions eventually grow faster than do quadratic functions.

Lesson Narrative

In this lesson, students investigate how quantities that grow quadratically compare to those that grow exponentially. They discover and reason that increasing exponential functions eventually surpass increasing quadratic functions. By examining successive quotients for each type of function, students see that the outputs of quadratic functions are not multiplied by the same factor each time the input increases by one. In fact, these successive quotients get smaller as the inputs increase, while the outputs of the exponential functions continue to have the same multiplier. As they compare the two types of functions, they develop their understanding of quadratic expressions and how the shape of the graph differs between the two types of functions.

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. We recommend making technology available.

Standards

Building On	**6.EE.A.1**
Addressing	**HSF-BF.A.1.a, HSF-IF.C, HSF-LE.A.3**
Building Toward	**HSF-LE.A.3**

Instructional Routines

- 5 Practices
- Poll the Class

Required Materials

Materials to Gather

- Graphing technology: Activity 2, Activity 3

Required Preparation

Activity 2:

Acquire devices that can run Desmos (recommended) or other graphing technology. (Desmos is available under Math Tools.)

Activity 3:

Acquire devices that can run Desmos (recommended) or other graphing technology. (Desmos is available under Math

Tools.)

Student Facing Learning Goals

Let's compare quadratic and exponential changes and see which one grows faster.

4.1 From Least to Greatest

5 min

Warm-up

Activity Narrative

In this *Warm-up*, students compare the values of exponential expressions, by making use of their structure (MP7). The reasoning here prepares them to think about exponential growth later in the lesson.

Students should recognize that $9^2 < 10^2$ and $2^9 < 2^{10}$. Deciding whether 10^2 or 2^9 is greater requires some estimation or further reasoning using properties of exponents.

For example, some students may recognize that $2^4 = 16$ and $2^8 = 2^4 \cdot 2^4 = \left(2^4\right)^2$, so $2^8 = 16^2$, which is 256. Because 2^9 is greater than 2^8, it follows that 2^9 is greater than 256 and therefore greater than 10^2.

As students discuss their thinking, listen for strategies that involve using properties of exponents or thinking about the structure of the expressions.

Standards

Building On	**6.EE.A.1**
Building Toward	**HSF-LE.A.3**

Launch

Arrange students in groups of 2. Give students a moment of quiet think time and then time to share their thinking with a partner.

In order to encourage students to rely on the structure of the expressions, they should not use a calculator to evaluate the expressions in this activity.

Student Task Statement

List these quantities in order, from least to greatest, without evaluating each expression. Be prepared to explain your reasoning.

2^{10} 10^2 2^9 9^2

Student Response

$9^2, 10^2, 2^9, 2^{10}$

Activity Synthesis

Select students to share their responses and reasoning. Highlight explanations that show that the expressions can be compared by analyzing their structure (as in the example in the *Activity Narrative*), and that it is not necessary to know their exact values to put the expressions in order.

15 min

4.2 Which One Grows Faster?

Activity Narrative

This activity prompts students to contrast quantities that grow exponentially and quadratically, by writing equations and creating tables of values. Before students begin working, they are asked to make an estimate of the number of squares in each pattern at Step 5 and Step 10. Making a reasonable estimate and comparing a computed value to one's estimate is often an important aspect of making sense of problems (MP1). Later from the tables, students notice that the output of the exponential function eventually outgrows that of the quadratic function. In the next activity, they will think further about whether this is always the case.

If students opt to use spreadsheet or graphing technology, they practice choosing appropriate tools strategically (MP5).

Standards

Addressing **HSF-BF.A.1.a, HSF-LE.A.3**

Instructional Routines

- Poll the Class

Launch

Display the image of the patterns for all to see, and ask students to read the description of how the patterns grow. Ask students to predict which pattern will have more small squares in Step 5.

Then, ask students to predict which pattern will have more small squares in Step 10. Poll the class to collect their predictions. Display the number of students who think pattern A will have more small squares and the number who think pattern B will have more small squares.

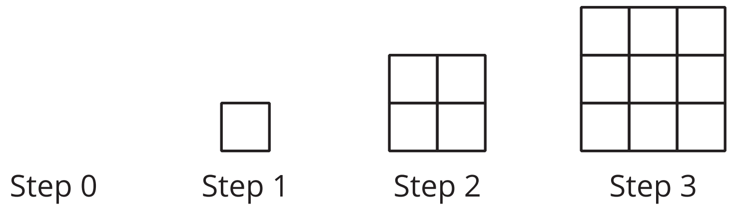
Arrange students in groups of 2.

Some students may choose to use a spreadsheet tool to study the pattern, and subsequently to use graphing technology to plot the data. Others may wish to use a calculator to compute growth factors. Provide access to graphing technology, a scientific calculator, or devices that can run a spreadsheet tool.

Student Task Statement

- In Pattern A, the length and width of the rectangle grow by one small square from each step to the next.
- In Pattern B, the number of small squares doubles from each step to the next.
- In each pattern, the number of small squares is a function of the step number, n.

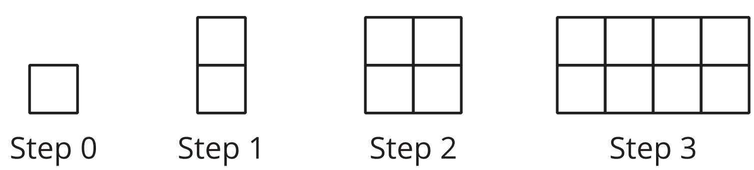
Pattern A



1. Write an equation to represent the number of small squares at Step n in Pattern A.
2. Is the function linear, quadratic, or exponential?
3. Complete the table:

n, step number	$f(n)$, number of small squares
0	
1	
2	
3	
4	
5	
6	
7	
8	

Pattern B



1. Write an equation to represent the number of small squares at Step n in Pattern B.
2. Is the function linear, quadratic, or exponential?

3. Complete the table:

n, step number	$g(n)$, number of small squares
0	
1	
2	
3	
4	
5	
6	
7	
8	

How would the two patterns compare if they continue to grow? Make 1–2 observations.

Student Response

1. For Pattern A, $f(n) = n^2$ and for Pattern B, $g(n) = 2^n$.
2. Function f is quadratic, and function g is exponential.
3. See the tables in the *Activity Synthesis*.

Sample observations: The number of small squares in Pattern B grows much more quickly than the number of squares in Pattern A grows, once n is greater than 5. The patterns have the same number of squares when $n = 2$ and when $n = 4$.

Building on Student Thinking

Some students may write the equation for pattern B as $g(n) = 2n$. Point out that pattern B is *doubling* the number of small squares. Step 3 would have 8 small squares. Prompt students to test their equation when $n = 3$ to see if it gives the correct output. $g(3) = 2(3) = 6$ not 8, so a linear function does not work. Since pattern B is doubling, the function is exponential not linear. A linear pattern such as $2n$ would add 2 small squares at each step rather than double the number of small squares.

Activity Synthesis

Select students to share their equations and to display their tables for all to see. Invite others to share their observations about the values in the tables.

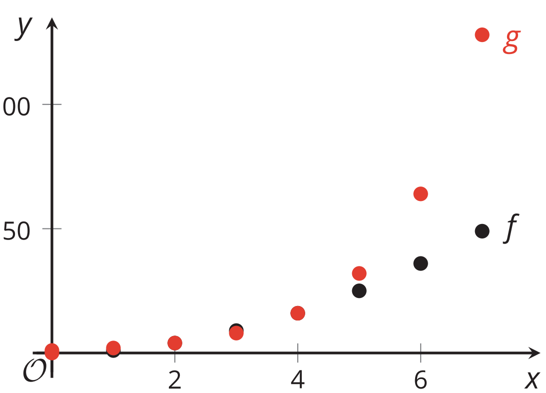
To help students understand why the value of the exponential function outgrows that of the quadratic function, consider showing tables that contrast the output values of f and g and amending each with a third column that shows their growth factors as n goes up by 1.

n, step number	$f(n)$, number of squares	growth factor (to 2 places)
0	0	
1	1	undefined
2	4	$\frac{4}{1} = 4$
3	9	$\frac{9}{4} = 2.25$
4	16	$\frac{16}{9} = 1.78$
5	25	$\frac{25}{16} = 1.56$
6	36	$\frac{36}{25} = 1.44$
7	49	$\frac{49}{36} = 1.36$
8	64	$\frac{64}{49} = 1.31$

n, step number	$g(n)$, number of squares	growth factor
0	1	
1	2	$\frac{2}{1} = 2$
2	4	$\frac{4}{2} = 2$
3	8	$\frac{8}{4} = 2$
4	16	$\frac{16}{8} = 2$
5	32	$\frac{32}{16} = 2$
6	64	$\frac{64}{32} = 2$
7	128	$\frac{128}{64} = 2$
8	256	$\frac{256}{128} = 2$

Highlight the fact that a fundamental feature of an exponential function is that it changes by equal factors over equal intervals. In this exponential function, the output increases by a factor of 2 at each step.

In the quadratic function, we can see that the output changes by a factor of 4, then $2\frac{1}{4}$, then $1\frac{7}{9}$, and so on. Even though it starts out growing faster than the exponential function is growing, the growth factor of the quadratic function decreases at each step and falls below 2 after a couple of steps. In the meantime, the growth factor of the exponential function stays at 2.



Also consider showing the graphs representing the two functions, to help students visualize the data in the tables. This graph shows the outputs of f and g, at whole-number inputs.

Discuss how the graphs representing both quadratic and exponential functions curve upward. The two are very close together for small values of x. As x continues to grow, however, the values of g become much greater than those of f and continue to increase more quickly.

Access for Students with Disabilities

Representation: Internalize Comprehension. Use color coding and annotations to highlight connections between

representations in a problem. For example, color code the connections between the growth factor and the graph of quadratic and exponential functions.
Supports accessibility for: Visual-Spatial Processing

4.3 Comparing Two More Functions

15 min

Activity Narrative

Students continue to compare quadratic and exponential functions in this activity. This time, they decide how to compare the functions.

Monitor for students who use different strategies to compare the functions.

- Create one or more tables of values, and compare the values of $p(x)$ and $q(x)$ at increasingly large values of x.
- Graph the functions defined by $p(x) = 6x^2$ and $q(x) = 3^x$, and compare the graphs.
- Create one or more tables of values, calculate the growth factors at equal intervals of input, and then compare the growth factors.

Have students present in this order to support moving them from checking large values for these particular functions toward noticing a pattern that can apply to more general quadratic and exponential functions.

Making graphing or spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Addressing **HSF-IF.C, HSF-LE.A.3**

Instructional Routines

- 5 Practices

Launch

Ask students to observe the equations representing the two functions and to determine which function is exponential and which is quadratic. Invite students to share how they know. Make sure students recognize that p is quadratic and q is exponential before they begin the activity.

Provide access to graphing technology and spreadsheet tools. It is ideal if each student has their own device. This may be a good opportunity for students to experiment with the graphing window. If the horizontal dimension is very small (for example, $0 < x < 5$) or the vertical dimension is very large (for example, $0 < y < 3{,}000$), the two graphs will be hard to distinguish. As needed, remind students to think about adjusting the graphing window to make the graphs more informative.

Select students with different strategies, such as those described in the *Activity Narrative*, to share later.

Access for Students with Disabilities

Action and Expression: Provide Access for Physical Action. Support effective and efficient use of tools and assistive technologies. To use graphing technology, some students may benefit from a checklist or list of steps that help them adjust the graphing window to experiment with the horizontal and vertical dimensions.
Supports accessibility for: Organization, Memory, Attention

Student Task Statement

Here are two functions: $p(x) = 6x^2$ and $q(x) = 3^x$.

Investigate the output of p and q for different values of x. For large enough values of x, one function will have a greater value than the other. Which function will have a greater value as x increases?

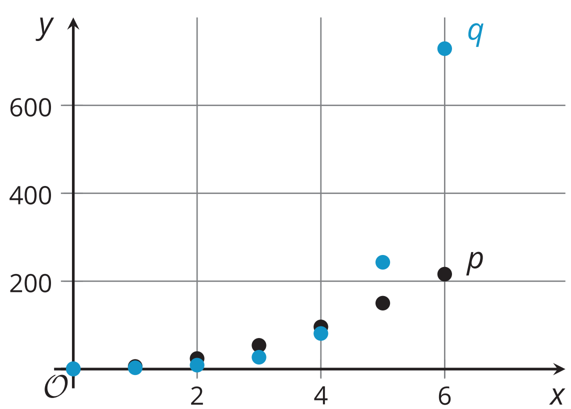
Support your answer with tables, graphs, or other representations.

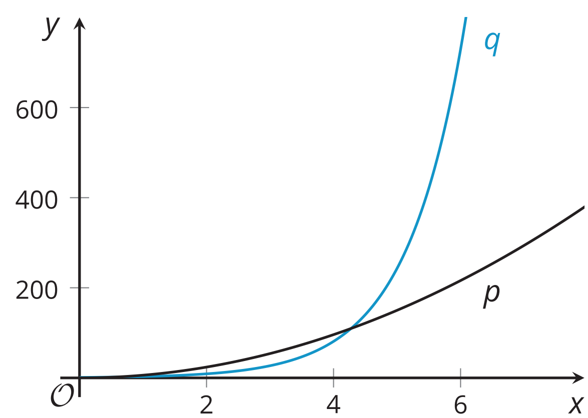
Student Response

Sample response: q is larger when $x = 0$ and again when $x = 5$. Once x is larger than 5, q is growing much more quickly, as shown in the table.

x	$p(x)$	$q(x)$
0	0	1
1	6	3
2	24	9
3	54	27
4	96	81
5	150	243
6	216	729
7	294	2,187

Alternatively, here is a graph showing the plotted values from the table, and another showing $y = p(x)$ and $y = q(x)$.





Are You Ready for More?

1. Jada looks at the function $g(x) = 1.1^x$ and says that this exponential function grows more slowly than the quadratic function $f(x) = x^2$. Do you agree with Jada? Explain your reasoning.
2. Let $f(x) = x^2$. Could you have an exponential function $g(x) = b^x$ so that $g(x) < f(x)$ for all values of x?

Extension Student Response

1. No. It takes a while for the exponential function to catch up to the quadratic, but for any $x \geq 96$ the exponential function is greater than the quadratic function.
2. No, $g(0) > f(0)$ for all values of b (except 0), because $b^0 = 1$ and $0^2 = 0$. (If $b = 0$, there are problems when $x \leq 0$, so this is not usually considered an exponential function.)

Activity Synthesis

Invite previously selected students to share their strategies for comparing the functions. Sequence the discussion of the strategies by the order listed in the *Activity Narrative*. If possible, record and display their work for all to see.

If no students chose to graph the functions, consider displaying the graphs for all to see.

Connect the different responses to the learning goals by asking questions such as:

- "What technology is helpful in comparing the growth of the functions?" (Spreadsheets or calculators can help find the factors between consecutive integer inputs. Graphing technology is useful for seeing which graph has greater values when looking at them with the right window.)
- "How do these functions compare to the growth of $r(x) = 2^x$?" (r eventually grows faster than p, but grows slower than q for positive x values.)
- "Recall that $\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2$. How does this help to show that the factors for quadratic functions get close to 1 with large values of x?" (Compare the values when the inputs are 100 and 101. $\frac{101^2}{100^2} = \left(\frac{101}{100}\right)^2 = (1.01)^2$ which is close to 1. For greater values, this ratio gets even closer to 1.)

Lesson Synthesis

Ask students to reflect on how they analyzed the behaviors of quadratic and exponential functions. Discuss questions such as:

- "What are some ways for comparing quadratic growth and exponential growth?" (By comparing their values in a table, by graphing the equations representing them, or by comparing the growth factors as the input increases.)
- "When we compared n^2 and 2^n, we saw the value of 2^n become greater than n^2 at $n = 5$. When we compared $6x^2$ and 3^x, we saw 3^x overtaking $6x^2$ by the time x reaches 5. If we compare, say, $1{,}000x^2$ and 2^x, will the exponential still overtake the quadratic? If so, at what x value do you think it would happen? If not, why not?" (Yes. It'd probably happen when x is between 15 and 20.)

4.4 Comparing $5x^2$ and 2^x

Cool-down

5 min

Standards

Addressing HSF-LE.A.3

Student Task Statement

Tyler completes the table comparing values of the expressions $5x^2$ and 2^x.

x	$5x^2$	2^x
1	5	2
2	20	4
3	45	8
4	80	16

Tyler concludes that $5x^2$ will always be greater than 2^x for the same value of x. Do you agree? Explain or show your reasoning.

Student Response

Tyler is not correct. For small values of x, $5x^2$ is greater than 2^x, but 2^x is eventually greater because it always doubles when x increases by 1. When $x = 9$, $5(9)^2 = 405$, and $2^9 = 512$, so 2^x is larger.

Responding to Student Thinking

Points to Emphasize
If most students struggle with comparing exponential and quadratic expressions, revisit this skill as opportunities arise. For example, invite 2–3 students to share their responses to the Practice Problem referred to here during class time.

Integrated Math 2, Unit 4, Lesson 6, Practice Problem 7

Lesson 4 Summary

The graphs of quadratic functions and the graphs of exponential functions with a base that is greater than 1 both curve upward. To compare the two, let’s look at the quadratic expression $3n^2$ and the exponential expression 2^n.

A table of values shows that $3n^2$ is initially greater than 2^n, but 2^n eventually becomes greater.

n	$3n^2$	2^n
1	3	2
2	12	4
3	27	8
4	48	16
5	75	32
6	108	64
7	147	128
8	192	256

Here's why exponential growth eventually overtakes quadratic growth.

- When n increases by 1, the exponential expression 2^n always increases by a factor of 2.
- The quadratic expression $3n^2$ increases by different factors, depending on n, but these factors get smaller. For example, when n increases from 2 to 3, the factor is $\frac{27}{12}$ or 2.25. When n increases from 6 to 7, the factor is $\frac{147}{108}$ or about 1.36. As n increases to larger and larger values, $3n^2$ grows by a factor that gets closer and closer to 1.

In general, quadratic functions change with a factor that gets closer and closer to 1 as the input to the function gets larger. Exponential functions always grow with the same factor, so if that growth factor is greater than 1, then the exponential function will eventually grow faster than any quadratic function will.

Lesson 4 Practice Problems

1 Student Task Statement

The table shows values of the expressions $10x^2$ and 2^x.

a. Describe how the values of each expression change as x increases.

b. Complete the table.

c. Make an observation about how the values of the two expressions change as x becomes greater and greater.

x	$10x^2$	2^x
1	10	2
2	40	4
3	90	8
4	160	16
8		
10		
12		

Solution

a. Sample response: $10x^2$ grows by increasing amounts but by decreasing growth factors each time x increases by 1. The expression 2^x is changing exponentially. It doubles each time x increases by 1.

b. For $10x^2$ the values are 640, 1,000, and 1,440. For 2^x the values are 256, 1,024, and 4,096.

c. Sample response: The value of 2^x grows much more quickly as x increases. When x is 10, the values of the two expressions are pretty close, but when x is 12, the value of the exponential expression is almost 3 times that of the quadratic expression.

2 Student Task Statement

Function f is defined by $f(x) = 1.5^x$. Function g is defined by $g(x) = 500x^2 + 345x$.

a. Which function is quadratic? Which one is exponential?

b. The values of which function will eventually be greater for larger and larger values of x?

Solution

a. f is exponential and g is quadratic.

b. The exponential function, f.

3 Student Task Statement

Create a table of values to show that the exponential expression $3(2)^x$ eventually overtakes the quadratic expression $3x^2 + 2x$.

Solution

Sample response:

When $x = 5$, the value of $3(2)^x$ is greater than the value of $3x^2 + 2x$.

x	$3(2)^x$	$3x^2 + 2x$
1	6	5
2	12	16
3	24	33
4	48	56
5	96	85

4

Student Task Statement

The table shows the values of 4^x and $100x^2$ for some values of x.

Use the patterns in the table to explain why eventually the values of the exponential expression 4^x will overtake the values of the quadratic expression $100x^2$.

x	4^x	$100x^2$
1	4	100
2	16	400
3	64	900
4	256	1600
5	1024	2500

Solution

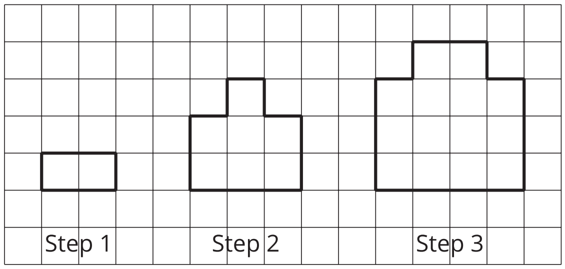
Sample response: The value of the quadratic expression grows by different factors each time x goes up by 1. When x increases from 1 to 2, $100x^2$ grows by a factor of 4. When x goes from 2 to 3, $100x^2$ grows by a factor of 2.25. Next, it grows by a factor of about 1.78. Each time x increases by 1, $100x^2$ grows by a smaller factor than the previous time. In contrast, the value of 4^x grows by a factor of 4 every time x goes up by 1, so eventually, it will catch up to the value of the quadratic expression.

5

from Unit 4, Lesson 2

Student Task Statement

Here is a pattern of shapes. The area of each small square is 1 sq cm.



a. What is the area of the shape in Step 10?
b. What is the area of the shape in Step n?
c. Explain how you see the pattern growing.

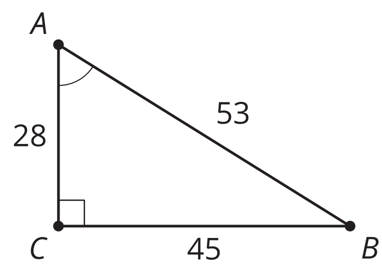
Solution

a. 119 sq cm
b. $(n+1)^2-2$ sq cm (or equivalent).
c. Each figure is a square whose side length is one greater than the step number, without the two small corner squares. In step 10, the figure would be an 11 cm by 11 cm square without the two corner squares.

6 from Unit 3, Lesson 10

Student Task Statement

Select **all** expressions that give the measure of angle A.



A. $\arccos\left(\frac{28}{53}\right)$
B. $\arccos\left(\frac{45}{53}\right)$
C. $\arcsin\left(\frac{28}{53}\right)$
D. $\arcsin\left(\frac{45}{53}\right)$
E. $\arctan\left(\frac{28}{45}\right)$
F. $\arctan\left(\frac{45}{28}\right)$

Solution

A, D, F

