Unit 4, Lesson 8

Unknown Exponents

Goals

- Comprehend the need to "undo" exponentiation to solve problems in simple contexts.
- Explain (orally and in writing) how to determine the value of an unknown exponent both exactly and approximately.

Learning Targets

- I can approximate the value of unknown exponents.

Lesson Narrative

In this lesson, students recognize the need to find an unknown exponent in an exponential expression or equation. First, students are given a visual pattern that shows a geometric sequence and asked to find where in the sequence that a certain value can be found. Then, students use an exponential model for population growth to estimate when the population will reach certain values.

An optional practice activity prompts students to find the exponent for an expression of the form 2^x or 5^x that would result in a given value.

To find the values of exponents, students must make sense of the problem and persevere in finding an estimate for the values (MP1).

Technology isn't required for this lesson, but there are opportunities for students to choose to use appropriate technology to solve problems. We recommend making technology available.

Standards

Building On	**HSA-REI.A.2, HSA-REI.B.4.b, HSF-LE.A.2**
Addressing	**HSF-BF.A.1.a**
Building Toward	**HSF-LE.A.4**

Instructional Routines

- MLR7: Compare and Connect
- MLR8: Discussion Supports

Required Materials

Materials to Gather

- Scientific calculators: Activity 3, Activity 4, Cool-down

Required Preparation

Activity 2:

Provide access to spreadsheet or graphing technology.

Activity 3:

Provide access to graphing technology.

Student Facing Learning Goals

Let's find unknown exponents.

8.1 A Bunch of x's

Warm-up

5 min

Activity Narrative

This activity reminds students that they have learned to solve equations involving various operations, and that the process of solving an equation involves maintaining the equality of each statement and thinking about what value of x keeps the equation true. This work prepares students to think about how to solve equations when the unknown value is an exponent.

Standards

Building On **HSA-REI.A.2, HSA-REI.B.4.b**

Student Task Statement

Solve each equation. Be prepared to explain your reasoning.

1. $\frac{x}{3} = 12$
2. $3x^2 = 12$
3. $x^3 = 12$
4. $\sqrt[3]{x} = 12$
5. $\sqrt{3x} = 12$
6. $\frac{3}{x} = 12$

Student Response

1. $x = 36$
2. $x = 2$ or -2
3. $x = \sqrt[3]{12}$ (or equivalent)
4. $x = 12^3$ (or equivalent)
5. $x = 48$
6. $x = \frac{1}{4}$

Activity Synthesis

Invite students to share their responses and reasoning. Point out the different ways in which students found the value of an unknown variable in equations (such as relying on a familiar math fact, working backward, or performing an inverse operation).

Tell students that in this lesson we will look at some equations where the value we want to find is an exponent.

15 min

8.2 A Tessellated Trapezoid

Activity Narrative

This activity motivates a need to find exponents to solve problems in a discrete geometric context. Students are given a visual pattern in which the number of objects increases exponentially, then asked to find the step in the pattern when a certain number of objects would appear.

Monitor for students who use these strategies:

- Keep multiplying by 4 and count the number of times it takes to reach 262,144.
- Use a calculator to evaluate 4^n for different values of n until it equals 262,144.
- Reason about known values and skip ahead in the pattern more than 1 step at a time.
- Create a table of values or a spreadsheet (with the step number being the input and 4^n being the output) and extend the table until the output is 262,144.
- Write an equation for $y = 4^x$, graph it, and use the graph to see where it reaches a y value of 262,144.

Making graphing or spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

Access for English Language Learners

This activity uses the *Compare and Connect* math language routine to advance representing and conversing as students use mathematically precise language in discussion.

Standards

Building On	**HSF-LE.A.2**
Addressing	**HSF-BF.A.1.a**
Building Toward	**HSF-LE.A.4**

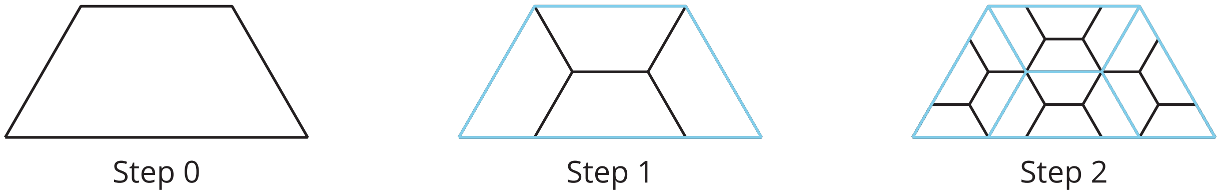
Instructional Routines

- MLR7: Compare and Connect

Launch

Display the image of the three steps for all to see. Ask students how many small trapezoids are in each of the steps $(1, 4, 16)$ and how they counted. Then ask students to describe the pattern they notice in the image as well as in the sequence of numbers.

If nobody points it out, share this image, highlighting the relationship to the previous step in the visual pattern.



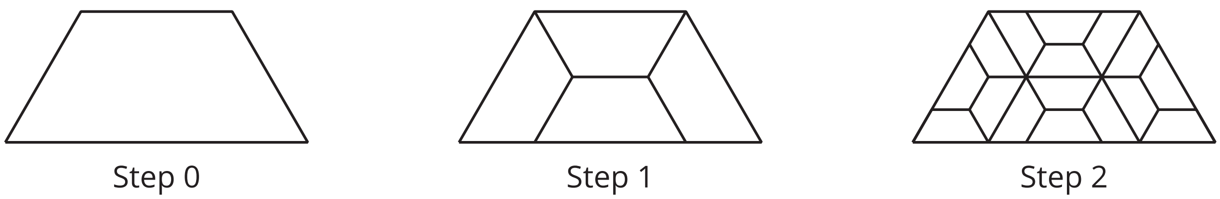
Select work from students who used different strategies, such as those described in the *Activity Narrative*, to share later.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support development of organizational skills in problem-solving, chunk this task into more manageable parts. For example, check in with students after 2–3 minutes of work time to share their thinking about the connections between the step number and the number of trapezoids in that step.
Supports accessibility for: Organization, Attention

Student Task Statement

Here is a pattern showing a trapezoid decomposed into similar trapezoids at each step.



1. If n is the step number, how many of the smallest trapezoids are there when n is 4? What about when n is 7?
2. At a certain step, k, there are 262,144 smallest trapezoids.
    a. Write an equation to represent the relationship between k and the number of trapezoids in step k.
    b. Explain to a partner how you might find the value of k.

Student Response

1. When n is 4, there are 4^4 or 256 trapezoids. When n is 7, there are 4^7 trapezoids.
2. a. $4^k = 262{,}144$, or $1 \cdot 4^k = 262{,}144$
    b. Sample response: I would keep multiplying by 4 and count the number of times it takes to reach 262,144.

Activity Synthesis

The goal of the discussion is to discuss ways to jump ahead in a sequence of values growing with an exponential pattern.

Display 2–3 strategies from previously selected students for all to see. If time allows, invite students to briefly describe their strategy, then use *Compare and Connect* to help students compare, contrast, and connect the different strategies. Here are some questions for discussion:

- "What do the strategies have in common? How are they different?"
- "Which strategies would you select if the number of trapezoids is small, like 64? Which strategies would you select if the number of trapezoids is large, like 268,435,456?"
- "Other than guessing, how can we solve the equation you wrote that uses k?"

8.3 Successive Splitting

15 min

Activity Narrative

In this activity, students find exponents in a continuous growth context, where the unknown exponents may not be integers.

Students may choose similar strategies for finding the value of an exponent as in the previous activity, but they will find that these strategies do not produce exact answers, making it necessary to make estimates and then check them, or to use alternative strategies.

In the first question, students are likely to use a calculator to evaluate the expression $100 \cdot 3^h$ for different values of h. In the second question, to estimate the time when the colony would reach a certain population, students may estimate using a calculator (perhaps repeatedly, to get increasingly precise answers) or estimate using a graph. Monitor for students who use these strategies, and invite them to share during the whole-class discussion.

Also monitor for students who can reason that the number of hours in the last question is less than 1 by reasoning concretely and abstractly (MP2):

- The population triples in 1 hour, so it must take less than 1 hour to double.
- $100 \cdot 3^h = 200$ is equivalent to $3^h = 2$. If h is 1, then the expression on the left has a value of 3, so h must be less than 1.

Students should consider what available tools would be most helpful for estimating the solutions (MP5).

Standards

Building Toward **HSF-LE.A.4**

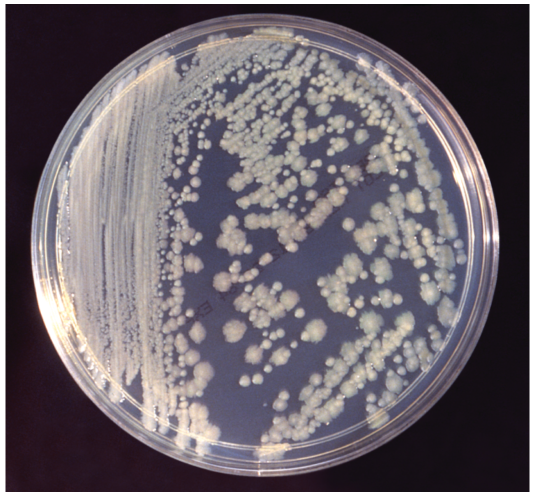
Instructional Routines

- MLR8: Discussion Supports

Launch

Arrange students in groups of 2. Provide access to scientific calculators. Ask students to read the *Task Statement* and to complete the first question. After a few minutes of quiet think time, ask them to share their answers and reasoning with their partner before selecting 2–3 students to share how they determined the population for the three different times.

Student Task Statement



In a lab, a colony of 100 thousand bacteria is placed on a petri dish. The population grows exponentially, tripling every hour.

1. How would you estimate or find the population of bacteria in:
    a. 4 hours?
    b. 90 minutes?
    c. $\frac{1}{2}$ hour?
2. How would you estimate or find the number of hours it would take the population to grow to:
    a. 1,000 thousand bacteria?
    b. double the initial population?

Student Response

1. Sample responses:
    a. Multiply 100 by 3 four times. Or, Find $100 \cdot 3^4$.
    b. Estimate that it will be a little less than halfway between 300 (or $100 \cdot 3^1$) thousand and 900 (or $100 \cdot 3^2$) thousand. Or, find $100 \cdot 3^{(1.5)}$ with a calculator.
    c. Find $100 \cdot 3^{\frac{1}{2}}$, which is $100 \cdot \sqrt{3}$, which is about 17 thousand. Or, create a graph for $p = 100 \cdot 3^t$ and estimate p when t is $\frac{1}{2}$.
2. Sample responses:
    a. Multiply 100 by 3 until it reaches or passes 1,000, and then approximate the time.
    Enter $100 \cdot 3^h$ in a calculator, and use different decimals between 2 and 3 for the exponent (the h) until the result is about 1,000. (After 2 hours there are 900 thousand bacteria in the colony. After 3 hours there are 2,700 thousand, so the time would be close to 2 hours.)
    Graph $p = 100 \cdot 3^x$, and estimate the value of x where the graph reaches 1,000.
    b. Enter $100 \cdot 3^h$ in a calculator, and use different decimals between 0.5 and 1 for the exponent (the h) until the result is about 200.
    Graph $p = 100 \cdot 3^x$, and estimate the value of x where the graph reaches 200.

Building on Student Thinking

If students are unsure of how to find the values for the population that are not whole numbers, consider saying:

- "Tell me more about how you found the population for 4 hours."
- "How could you use a calculator or graphing technology to estimate the values of the populations at different times?"

Are You Ready for More?

A $1,000 investment increases in value by 5% each year. About how many years does it take for the value of the investment to double? Explain how you know.

Extension Student Response

Sample response: Because $(1.05)^{14} \approx 1.98$ and $(1.05)^{15} \approx 2.08$ it takes a little more than 14 years but less than 15 years.

Activity Synthesis

Invite students to share their strategies for answering the second question. If no student mentions graphing as a strategy, bring it up and demonstrate, if needed.

Clarify the distinctions in how to approach the two sets of questions. Make sure students see that:

- In the first question, the population of the bacteria can be found more precisely because the exponent and the base are known. For example, the population of bacteria, p, after 5 hours can be found using $p = 100 \cdot 3^5$. The right side of this equation can be evaluated exactly.
- The second question is about the value of the exponent. For example, answering "In how many hours will the colony double?" means finding h in the equation: $200 = 100 \cdot 3^h$. This equation can be rewritten as $2 = 3^h$. There are multiple ways to find or approximate the solution, and some ways are more involved than others.

Heading into future lessons, a guiding question will be "Is there a better way to find the value of h than by approximating?"

Access for English Language Learners

MLR8 Discussion Supports. Revoice student ideas to demonstrate and amplify mathematical language use. For example, revoice the student statement "I would graph the function" as a question such as, "How would graphing the exponential function help you find the number of hours it would take the population to grow to 1,000 bacteria?"
Advances: Speaking, Representing

8.4

Missing Values

Optional

10 min

Activity Narrative

In earlier activities, students try to find unknown exponents in contextual problems. This activity allows students to reason about exponential equations such as $2^5 = ?$ and $2^? = 1024$ in a straightforward, decontextualized task. Consider using this activity if it is needed to further motivate the upcoming work on logarithms.

When completing a blank cell in the bottom row of a table, students may choose to write an expression with an exponent. Encourage students to evaluate the expression and write a numerical expression without an exponent. To complete a blank cell in the top row, students need to work backward and use their understanding of integer exponents. For example, to find x when 2^x equals 256, they need to think about what power of 2 has a value of 256.

Standards

Building Toward **HSF-LE.A.4**

Launch

Provide access to scientific calculators.

Student Task Statement

Complete the tables.

x			-1	0	$\frac{1}{2}$	1			5		
2^x	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{1}{2}$				4	16		256	1,024

x				$\frac{1}{3}$	$\frac{1}{2}$				
5^x	$\frac{1}{25}$	$\frac{1}{5}$	1			5	125	625	3,125

Be prepared to explain how you found the missing values.

Student Response

x	-5	-2	-1	0	$\frac{1}{2}$	1	2	4	5	8	10
2^x	$\frac{1}{32}$	$\frac{1}{4}$	$\frac{1}{2}$	1	$\sqrt{2}$	2	4	16	32	256	1,024

x	-2	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	1	3	4	5
5^x	$\frac{1}{25}$	$\frac{1}{5}$	1	$\sqrt[3]{5}$	$\sqrt{5}$	5	125	625	3,125

Building on Student Thinking

If students use a calculator to write approximate decimal answers for some of the roots, consider asking:

- "Can you explain how you used your calculator to find the missing values."
- "How could the sequence of buttons you used on the calculator help you write an expression that gives the exact value?"

Activity Synthesis

Invite students to share how they reasoned about the missing values in the top and bottom rows. Allow the class to hear as many strategies as time permits.

If any students used properties of exponents to reason about unknown exponents, highlight these strategies. For example, to reason about 1,024, students might say, "I know that 4 times 256 is 1,024. Because 4 is 2^2 and 256 is 16^2, which is $(2^4)^2$, or 2^8, then 1,024 is $2^2 \cdot 2^8$, which is 2^{10}."

Lesson Synthesis

Invite students to reflect on their process for finding an unknown input in an exponential function. Ask students questions like:

- "How is finding the step number that would produce a certain number of trapezoids similar to finding the hours when a bacteria population reaches 1,000 thousand?" (They both involve figuring out the value of an exponent.)
- "What are some ways to determine the exponent if we know the base and the value of the exponential expression? For example, if we know that 5^x has a value of 200 (or $5^x = 200$), how do we find or estimate x?" (We could:
    - Multiply by 5 repeatedly until the number reaches 200 or close to it, and count the number of times we multiplied.
    - Divide 200 by 5 repeatedly until it reaches 1, and count the number of times we divided.
    - Use a calculator to evaluate 5^x, trying different values of x until the result is (or approximates) 200.
    - Graph $y = 5^x$ and estimate where the graph has a y value of 200.)
- "How is the process of solving for an exponent like or unlike solving for a variable in an equation like $5x = 200$ or $x^5 = 200$?" (Like in solving other equations, sometimes we could see what value of x makes the equation true without doing too much. For example, we can tell that if $5^x = 25$, then x is 2. But other times, there's no quick way to "undo" the exponentiation or to isolate the x.)

Tell students that in upcoming lessons they will learn another way to find the value of an unknown exponent in exponential equations.

8.5 Video Viewers

Cool-down

5 min

Standards

Building Toward **HSF-LE.A.4**

Student Task Statement

On the day a video is posted online, 5 people watch the video. The next day the number of viewers doubles. Assume the number of viewers continues to double each day.

1. On which day will 640 people see the video? Explain or show your reasoning.
2. What strategy would you use to find the first day when more than 20,000 people will see the video (if the trend continues)?

Student Response

1. 7 days after it is posted. Sample reasonings:
   - Multiply 5 by 2 until it reaches 640.
   - Write $5 \cdot 2^d = 640$ and then $2^d = 128$, and then try different exponents to see what makes the power of 2 equal to 128.
2. Sample responses:
   - Multiply 5 by 2 repeatedly until the product is greater than 20,000.
   - Divide 20,000 by 2 until it is less than 5.
   - Divide 20,000 by 5 to get 4,000, and find the smallest power of 2 that is greater than 4,000.

Responding to Student Thinking

More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Lesson 8 Summary

Sometimes we know the value of an exponential expression but we don't know the exponent that produces that value.

For example, suppose the population of a town was 1 thousand. Since then, the population has doubled every decade and is currently at 32 thousand. How many decades has it been since the population was 1 thousand?

If we say that d is the number of decades since the population was 1 thousand, then $1 \cdot 2^d$, or just 2^d, represents the population, in thousands, after d decades. To answer the question, we need to find the exponent in $2^d = 32$. We can reason that since $2^5 = 32$, it has been 5 decades since the population was 1 thousand people.

When did the town have 250 people? Assuming that the doubling started before the population was measured to be 1 thousand, we can write: $2^d = 0.25$, or $2^d = \frac{1}{4}$. We know that $2^{-2} = \frac{1}{4}$, so the exponent d has a value of -2. The population was 250 two decades before it was 1,000.

But it may not always be so straightforward to calculate. For example, it is harder to tell the value of d in $2^d = 805$ or in $2^d = 4.5$. In upcoming lessons, we'll learn more ways to find unknown exponents.

Lesson 8 Practice Problems

1 Student Task Statement

A pattern of dots grows exponentially. The table shows the number of dots at each step of the pattern.

step number	0	1	2	3
number of dots	1	5	25	125

a. Write an equation to represent the relationship between the step number, n, and the number of dots, y.

b. At one step, there are 9,765,625 dots in the pattern. At what step number will that happen? Explain how you know.

Solution

a. $y = 5^n$

b. At step number 10. Sample reasonings:
- I keep multiplying by 5 and count the number of times until the result is 9,765,625.
- I use a calculator to evaluate 5^n at different values of n until I have 9,765,625.
- I graph $y = 5^n$ and then trace the graph to find the input that gives an output of 9,765,625.

2 Student Task Statement

A bacteria population is modeled by the equation $p(h) = 10{,}000 \cdot 2^h$, where h is the number of hours since the population was measured.

About how long will it take for the population to reach 100,000? Explain your reasoning.

Solution

Between 3 and 4 hours, but closer to 3 hours. Sample reasoning: The population doubles every hour, so doubling 3 times gives $10{,}000 \cdot 2^3$, or 80,000, and doubling 4 times gives $10{,}000 \cdot 2^4$, or 160,000.

3 Student Task Statement

Complete the table.

x			-2	0	$\frac{1}{3}$	1		
10^x	$\frac{1}{10{,}000}$	$\frac{1}{1{,}000}$	$\frac{1}{100}$				1,000	1,000,000,000

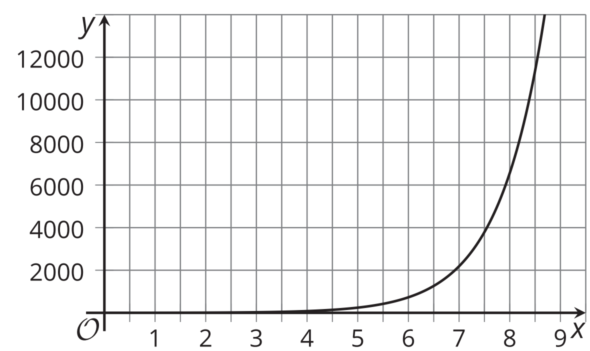
Solution

x	-4	-3	-2	0	$\frac{1}{3}$	1	3	9
10^x	$\frac{1}{10{,}000}$	$\frac{1}{1{,}000}$	$\frac{1}{100}$	1	$\sqrt[3]{10}$	10	1,000	1,000,000,000

4 Student Task Statement

Here is a graph of $y = 3^x$.

What is the approximate value of x satisfying $3^x = 10{,}000$? Explain how you know.



Solution

Sample response: About 8.4. The value is larger than 8 but smaller than 9. It looks like it is closer to 8 than to 9.

5 Student Task Statement

A person puts the same amount of money in two different savings accounts and leaves it to gain interest for a long time. The amount of money in one account doubles every 2 years. The amount of money in the second account triples every 3 years. Which account is growing more rapidly?

Solution

Sample response: After 6 years, the account that is doubling every 2 years increases by a factor of 8. The account that is tripling every 3 years increases by a factor of 9. So the account that is tripling is increasing at a faster rate.

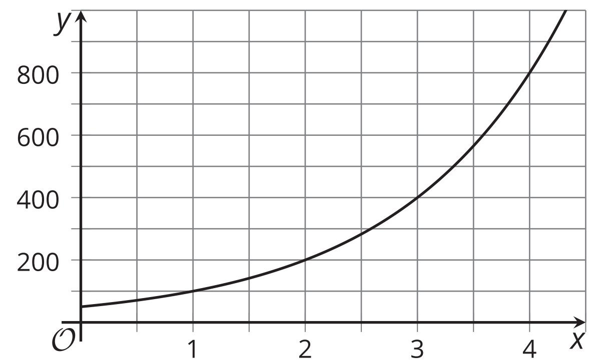
6 from Unit 4, Lesson 1

Student Task Statement

Describe how the output increases and the growth factor for these intervals.

a. inputs from 1 to 2

b. inputs from 3 to 4



Solution

Sample responses:

a. For inputs 1 to 2, the output increases by about 100 units, which is a growth factor of 2.

b. For inputs 3 to 4, the output increases by about 400 units, which is a growth factor of 2.

7 from Unit 4, Lesson 7

Student Task Statement

The half-life of carbon-14 is about 5730 years.

a. Complete the table, which shows the amount of carbon-14 remaining in a plant fossil at the different times since the plant died.

b. About how many years will it be until there is 0.1 picogram of carbon-14 remaining in the fossil? Explain how you know.

years	picograms
0	3
5730	
$2 \cdot 5730$	
$3 \cdot 5730$	
$4 \cdot 5730$	

Solution

a. 1.5, 0.75, 0.375, 0.1875

b. about $5 \cdot 5730$ or 29,000 years since there will be about 0.09375 picogram left after a fifth half-life