Unit 4, Lesson 4

# Representing Functions at Rational Inputs

Let's find how quantities are growing or decaying over fractional intervals of time.

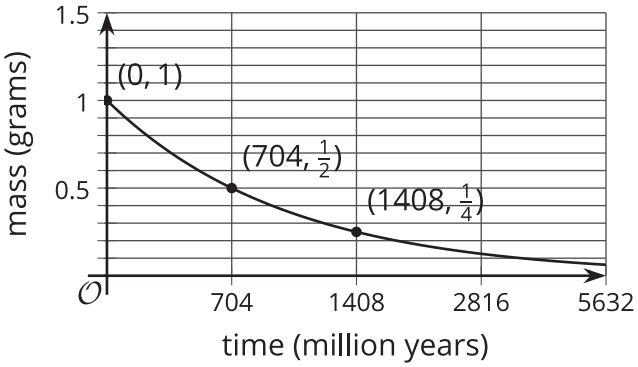
## 4.1 Radioactive Decay

Many materials such as radioactive elements or large molecules naturally break down over time at a rate that is proportional to how much of the material there is. The amount of material left over time is described mathematically using exponential decay.

To get a sense of how stable the materials are, the length of time it takes for half of the material to remain, or its **half-life**, is given. Here are some half-lives of a few radioactive materials and graphs of how much of the materials remain after there is 1 gram of the material.

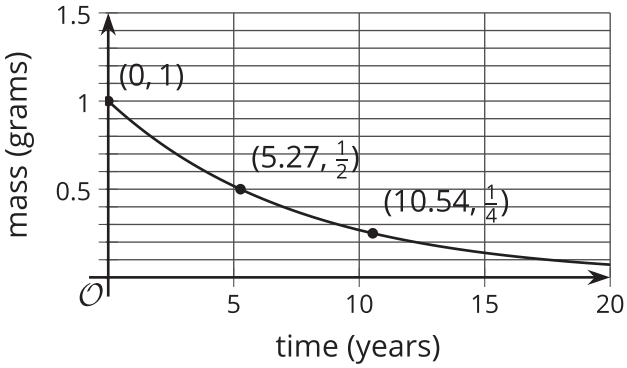
Uranium-235

Half-life: 704 million years




Cobalt-60

Half-life: 5.27 years

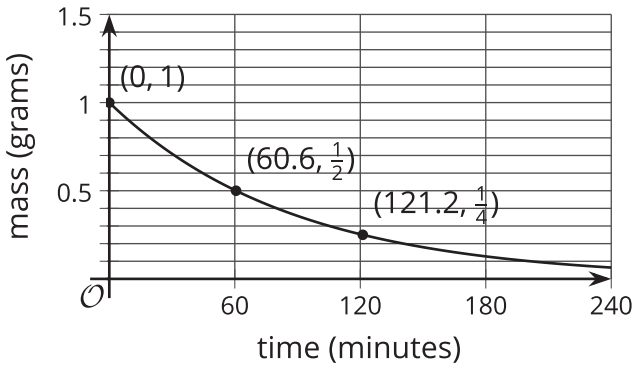



Bismuth-212

Half-life: 60.6 minutes

1. Which of these materials takes the longest to break down?

2. If you had a sample of 8 grams of cobalt-60, how long would it take for it to break down into only 1 gram left? Explain your reasoning.




## 4.2 Population of Nigeria



In 1990, Nigeria had a population of about 95.3 million. By 2000, there were about 122.4 million people, an increase of about 28.4%. During that decade, the population can be reasonably modeled by an exponential function.

1. Express the population of Nigeria $f(d)$, in millions of people, $d$ decades since 1990.

2. Write an expression to represent the population of Nigeria in 1996.

3. A student said, "The population of Nigeria grew at a rate of 2.84% every year because

$\frac{28.4}{10} = 2.84$."

a. Explain or show why the student's statement is incorrect.

b. Find the correct annual growth rate. Explain or show your reasoning.

# 4.3 Waiting for Waste

Cesium-137 is a radioactive material found in the waste of nuclear reactors. It has a half-life of about 30 years. Let's suppose that there are 100 grams of cesium-137 as part of some nuclear waste.

1. Each of these expressions describes the amount of cesium-137 in the nuclear waste some number of years after it is produced. For each expression, find the number of years it would take to have that much cesium-137 left for this waste.
   a. $100 \cdot \left(\frac{1}{2}\right)^{1}$
   b. $100 \cdot \left(\frac{1}{2}\right)^{3}$
   c. $100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{30}}$
   d. $100 \cdot \left(\frac{1}{2}\right)^{t}$
2. a. Write a function $g$ to represent the amount of cesium-137 left in the waste, $y$ years after it is produced.

   b. The function $f$ represents the amount of cesium-137 left in the waste after $t$ 30-year periods. Write an equation to represent the function $f$.

   c. Explain why $g(30) = f(1)$.

## Are you ready for more?

What percentage of the initial amount of cesium-137 do you expect to break down in the first 15 years: less than 25%, exactly 25%, or more than 25%? Explain or show your reasoning.

## Lesson 4 Summary

Imagine a material has a **half-life** of 3 hours. This means that the amount of material left after 3 hours is half of what there was when it started. For example, if there are 200 milligrams of the material at noon, then at 3 o'clock there will be 100 milligrams left, and at 6 o'clock there will be 50 milligrams left.

If a scientist has 200 mg of the material, then the amount of material, in mg, can be modeled by the function $f(t) = 200 \cdot \left(\frac{1}{2}\right)^t$. In this model, $t$ represents a unit of time. Notice that the 200 represents the initial amount of material the scientist has. The number $\frac{1}{2}$ indicates that for every 1 unit of time, the amount of material is cut in half. Because the half-life is 3 hours, this means that $t$ must measure time in groups of 3 hours.

But what if we wanted to find the amount of material the scientist has each hour after taking it? We know there are 3 equal groups of 1 hour in a 3-hour period. We also know that because the material decays exponentially, it decays by the same factor in each of those intervals. In other words, if $b$ is the decay factor for each hour, then $b \cdot b \cdot b = \frac{1}{2}$, or $b^3 = \frac{1}{2}$. This means that over each hour, the medicine must decay by a factor of $\sqrt[3]{\frac{1}{2}}$, which can also be written as $\left(\frac{1}{2}\right)^{\frac{1}{3}}$. So if $h$ is time in hours since the scientist had 200 mg of the material, we can express the amount of material in mg, $g$, the scientist has as $g(h) = 200 \cdot \left(\sqrt[3]{\frac{1}{2}}\right)^h$, or $g(h) = 200 \cdot \left(\frac{1}{2}\right)^{\frac{h}{3}}$.