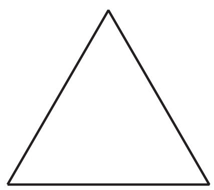
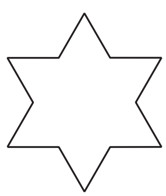
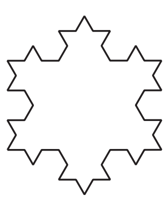
Unit 3, Lesson 10

# Summing Up

Let’s figure out a better way to add numbers.

## 10.1 Notice and Wonder: A Snowflake’s Return

What do you notice? What do you wonder?

| iteration | total number of triangles added since the first |
|---|---|
| **0**  | 0 |
| **1**  | 3 |
| **2**  | $3+3(4)=15$ |
| 3 | $3+3(4)+3(4)^2=63$ |
| $n$ | $3+3(4)+3(4)^2+\ldots+3(4)^{n-1}$ |

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## 10.2 A Geometric Addition Formula

Earlier, we learned that the $n^{\text{th}}$ term of a geometric sequence with an initial value of $a$ and a common ratio of $r$ is $a(r^{n-1})$.

For a Koch Snowflake, it turns out that we can find the number of triangles added on at each iteration by making $a = 3$ and $r = 4$. The sum $s$ of the first $n$ terms in this geometric sequence tell us how many triangles total make up the $n$th iteration of the snowflake

$$s = 3 + 3(4) + 3(4^2) + \ldots + 3(4^{n-1})$$

More generally, the sum of the first $n$ terms of any geometric sequence can be expressed as

$$s = a + a(r) + a(r^2) + \ldots + a(r^{n-1})$$

or

$$s = a(1 + r + r^2 + \ldots + r^{n-1})$$

1. What would happen if we multiplied each side of this equation by $(1 - r)$?
   (Hint: $(x - 1)(x^3 + x^2 + x + 1) = x^4 - 1$.)

2. Rewrite the new equation in the form of $s =$ ________.

3. Use this new formula to calculate how many triangles after the original are in the first 5, 10, and 15 iterations of the Koch Snowflake.

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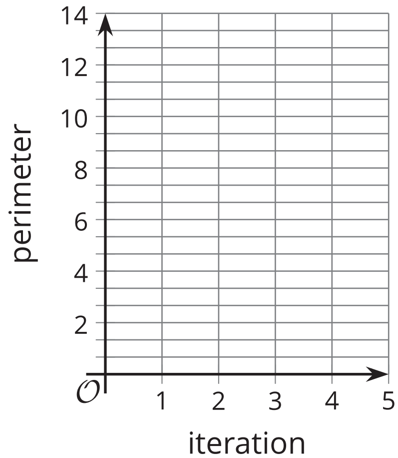
## Are you ready for more?

If the initial triangle has sides that are each 1 unit long, find an equation for the perimeter $P$ of the Koch Snowflake after the $n^{\text{th}}$ iteration, and graph $(n, P)$ for iterations 0 through 5.




## 10.3 Breakthrough Artist?

A music video is posted online and after a week it has 400,000 total views. The next day, the video has 13,000 new views, and each day following, the number of new views decreases by 12%.

1. How many people watched the video on the second, third, and fourth days after the decrease started?

2. How many total views will the video have after 28 days (21 days after the daily views started to decline)?

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## Lesson 10 Summary

Sometimes identities can help us see and write a pattern in a simpler form. Imagine a chessboard where 1 grain of rice is placed on the first square, 2 on the second, 4 on the third, and so on. How many grains of rice are on the 64-square chessboard? Trying to add up 64 numbers is difficult to do one at a time, especially because the first 20 squares have more than one million grains of rice on them! If we write out what this sum $s$ is, we have

$$s = 1 + 2 + 4 + \ldots + 2^{63}$$

If we rewrite this expression as $2^{63} + \ldots + 2^2 + 2 + 1$, we have an expression similar to one we've seen before, $x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$.

In an earlier lesson, we showed that $(x-1)(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1)$ is equivalent to the simpler expression $(x^n - 1)$. Using this identity with $x = 2$ and $n = 64$, we have

$$\begin{aligned} (2^{64-1} + 2^{64-2} + \ldots + 2^2 + 2 + 1) &= s \\ (2-1)(2^{64-1} + 2^{64-2} + \ldots + 2^2 + 2 + 1) &= (2-1)s \\ 2^{64} - 1 &= (2-1)s \\ \frac{2^{64} - 1}{2 - 1} &= s \\ 2^{64} - 1 &= s \end{aligned}$$

This means that the sum total of all the grains of rice is $2^{64} - 1$, or $18{,}446{,}744{,}073{,}709{,}551{,}615$. More generally, for any geometric sequence starting at $a$ with a common ratio $r$, the sum $s$ of the first $n$ terms is given by $s = a\frac{1-r^n}{1-r}$.