Unit 6, Lesson 10

Multiplying Complex Numbers

Goals

- Calculate the result of multiplying two complex numbers.
- Justify that two equivalent expressions involving complex numbers are equivalent, using the fact that $i^2 = -1$.

Learning Targets

- I can multiply complex numbers.

Lesson Narrative

This lesson continues to develop the idea that when complex numbers are combined, the result is also a complex number and can be written in the form $a + bi$, where a and b are real numbers. In this lesson, students use the fact that $i^2 = -1$ to multiply imaginary numbers, and use the strategies they develop to multiply complex numbers by writing the i^2 terms as real numbers.

Students make use of the familiar structure (MP7) of distributing terms to find the result of multiplying two complex numbers. To organize their thinking, they can use the same kind of diagrams they used in a previous unit to multiply polynomials. Students also take turns explaining their reasoning and critiquing the reasoning of others (MP3) as they match equivalent expressions: products of complex numbers and single values.

Standards

Building On	**HSN-RN.A.2**
Addressing	**HSN-CN.A.1, HSN-CN.A.2**
Building Toward	**HSN-CN.A.1, HSN-CN.A.2**

Instructional Routines

- MLR8: Discussion Supports
- Take Turns

Student Facing Learning Goals

Let's multiply complex numbers.

10.1 i Squared

Warm-up

5 min

Activity Narrative

The purpose of this activity is to elicit strategies and understandings students have for multiplying imaginary numbers. Later in this lesson, students will multiply complex numbers and write them in the form $a + bi$, so it will be helpful for students to see various strategies for multiplying imaginary numbers.

Standards

Building On	**HSN-RN.A.2**

Building Toward **HSN-CN.A.1, HSN-CN.A.2**

Student Task Statement

Write each expression in the form $a + bi$, where a and b are real numbers.

1. $4i \cdot 3i$
2. $4i \cdot -3i$
3. $-2i \cdot -5i$
4. $-5i \cdot 5i$
5. $(-5i)^2$

Student Response

1. $-12 + 0i$
2. $12 + 0i$
3. $-10 + 0i$
4. $25 + 0i$
5. $-25 + 0i$

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. Highlight ways that students regrouped factors of i or -1 in order to simplify their answers. If no student mentions the fact that squaring an imaginary number always results in a real number, ask students to discuss this idea.

10.2 Multiplying Imaginary Numbers

10 min

Activity Narrative

In this partner activity, students take turns matching equivalent expressions using the fact that $i^2 = -1$. They build on a previous activity in which they multiplied imaginary numbers and saw how this changed the numbers' representation on the complex plane. Fluency with strategically using the fact that $i^2 = -1$ will be an important skill in the next activity when students multiply numbers that have both real and imaginary parts. As students trade roles explaining their thinking and listening, they have opportunities to explain their reasoning and to critique the reasoning of others (MP3).

Standards

Addressing **HSN-CN.A.1**

Building Toward **HSN-CN.A.2**

Instructional Routines

- MLR8: Discussion Supports
- Take Turns

Launch

Arrange students in groups of 2. Tell students that for each expression in column A, one partner finds an equivalent expression in column B and explains why they think it is equivalent. (One item in column B will not be used.) The partner's job is to listen and make sure they agree. If they don't agree, the partners discuss until they come to an agreement. Students then swap roles. If necessary, demonstrate this protocol before students start working.

Access for English Language Learners

MLR8 Discussion Supports. Students should take turns finding a match and explaining their reasoning to their partner. Display the following sentence frame for all to see: "I noticed ________, so I matched ________." Encourage students to challenge each other when they disagree.
Advances: Speaking, Conversing, Representing

Student Task Statement

Take turns with your partner matching an expression in column A with an equivalent expression in column B.

- For each match that you find, explain to your partner how you know it's a match.
- For each match that your partner finds, listen carefully to the explanation. If you disagree, discuss your thinking and work to reach an agreement.

A	B
$5 \cdot 7i$	-9
$5i \cdot 7i$	$35i$
$3i^2$	-35
$(3i)^2$	1
$8i^3$	9
i^4	-3
$-i^2$	-1
$(-i)^2$	$-8i$

Student Response

1. $5 \cdot 7i = 35i$
2. $5i \cdot 7i = -35$
3. $3i^2 = -3$
4. $(3i)^2 = -9$
5. $8i^3 = -8i$
6. $i^4 = 1$

7. $-i^2 = 1$
8. $(-i)^2 = -1$

Building on Student Thinking

If students get stuck with the expressions i^3 or i^4, consider saying:

- "What powers of i do you already know a value for?"
- "How could writing out the repeated factors, for example, writing i^3 as $i \cdot i \cdot i$, help you to match i^3 with an equivalent expression?"

Activity Synthesis

Once all groups have completed the matching, here are some questions for discussion:

- "Which matches were tricky? Explain why." ($(-i)^2$ was tricky because there were several negative signs to keep track of and I had to double-check whether the final answer was negative or positive.)
- "Did you need to make adjustments in your matches? What might have caused an error? What adjustments were made?" (I thought $3i^2$ was -9 because I thought the 3 was squared too. I had to go back and just square the i and then multiply it by 3.)

Ask groups to explain their reasoning for several matches, especially why $i^4 = 1$. If not brought up by students, make sure to discuss that it's possible to use the fact that $i^2 = -1$ to make equivalent expressions.

10.3 Multiplying Complex Numbers

20 min

Activity Narrative

The purpose of this activity is for students to build fluency expressing the product of two complex numbers in the form $a + bi$, where a and b are real numbers. In order to do this, students must use the fact that $i^2 = -1$.

Look for students who answer the last question as $13 + 0i$ and students who answer 13, and highlight these during discussion.

Standards

Addressing **HSN-CN.A.1, HSN-CN.A.2**

Launch

Tell students that they are now going to multiply complex numbers together. Display the expression $(3 + 2i)(-4 - 5i)$ for all to see and give students 1 minute of quiet think time to consider how they would find the product. Invite a student to share a strategy, then ask if anyone else had a different strategy to share. If this strategy is not shared, display this table for all to see:

	3	$2i$
-4	-12	$-8i$
$-5i$	$-15i$	$-10i^2$

After a brief time to consider the diagram, select students to explain how they understand the table. Ask students, "Now that we have $-12 - 8i - 15i - 10i^2$, what do we do in order to write the number in the form $a + bi$?" (We know $i^2 = -1$, so the sum of these is $-12 - 8i - 15i + 10 = -2 - 23i$.)

Access for Students with Disabilities

Representation: Internalize Comprehension. Use color coding and annotations to highlight connections between representations in a problem. For example, color code real and imaginary parts of the complex numbers and the corresponding parts of the table.
Supports accessibility for: Visual-Spatial Processing

Student Task Statement

Write each product in the form $a + bi$, where a and b are real numbers.

1. $(8 + i)(-5 + 3i)$
2. $(-3 + 9i)(5i)$
3. $(3 + 2i)^2$
4. $(3 + 2i)(3 - 2i)$

Student Response

1. $-43 + 19i$
2. $-45 - 15i$
3. $5 + 12i$
4. $13 + 0i$ or just 13

Are You Ready for More?

On October 16, 1843, while walking across the Broom Bridge in Dublin, Ireland, Sir William Rowan Hamilton came up with an idea for numbers that would work sort of like complex numbers. Instead of just the number i (and its opposite $-i$) squaring to give -1, he imagined three numbers i, j, and k (each with an opposite) that squared to give -1.

The way these numbers multiplied with each other was very interesting. i times j would give k, j times k would give i, and k times i would give j. But the multiplication he imagined did not have a commutative property. When those numbers were multiplied in the opposite order, they'd give the opposite number. So j times i would give $-k$, k times j would give $-i$, and i times k would give $-j$. A *quaternion* is a number that can be written in the form $a + bi + cj + dk$ where a, b, c, and d are real numbers.

Let $w = 2 + 3i - j$ and $z = 2i + 3k$. Write each given expression in the form $a + bi + cj + dk$.

1. $w + z$

2. wz
3. zw

Extension Student Response

1. $2 + 5i - j + 3k$
2. $-6 + i - 9j + 8k$
3. $-6 + 7i + 9j + 4k$

Activity Synthesis

The key takeaway is that the product of complex numbers is another complex number, and we can see this by using usual arithmetic along with the fact that $i^2 = -1$ to write products in the form $a + bi$, where a and b are real numbers.

Select previously identified students to share their responses to the last question. Discuss the idea that numbers like 13 or $5i$ don't need to be written as $13 + 0i$ and $0 + 5i$ in order to be recognizable as complex numbers. Writing complex numbers as a single term is okay; it's something they did for a long time before they knew that all real numbers are complex numbers of the form $a + bi$ where $b = 0$. It can be helpful to be flexible in writing numbers in either format. For example, when using the strategy to multiply complex numbers using tables, the question here using $5i$ will match the other questions better if it is first written as $0 + 5i$.

If there is time, it can be helpful to compare multiplying complex numbers to multiplying two-digit numbers. Display the expression $23 \cdot 45$ and invite students to share how they would do the multiplication without a calculator. Display their thinking for all to see. Most methods will involve some form of partial products such as $20 \cdot 40$, $3 \cdot 40$, $20 \cdot 5$, and $3 \cdot 5$, which result from distributing from the expression $(20 + 3)(40 + 5)$. Point out that these partial products can be understood as "8 hundreds, 12 tens, 10 tens, and 15 ones." To find the sum of these to get the actual product of $23 \cdot 45$, we might need to rewrite some of these values in other ways, such as rewriting 12 tens as 1 hundred and 2 tens. Therefore the sum of the partial products is 10 hundreds, 3 tens, and 5 ones or 1,035.

The whole process is similar to finding a product like $(2 + i)(3 - 4i)$. First, we find the partial products: $2 \cdot 3, i \cdot 3, 2 \cdot -4i, i \cdot -4i$. Then, we rewrite the last partial product to find the sum. $6 + 3i - 8i - 4i^2 = 6 + 3i - 8i + 4 = 10 - 5i$.

Lesson Synthesis

The purpose of this discussion is for students to articulate how to multiply complex numbers. Display this equation for all to see, leaving room to record student thinking for all to see during the discussion:

$$(2 + 2i)(2 - 3i) = (3 + 2i) + (7 + 4i)$$

Ask students, "Mentally, without going through all the trouble of writing each side in the form $a + bi$, what about the structure of the expressions on each side of the equal sign tells you that the equation is true or false?" (Looking at the left-hand side, we can make the imaginary part by adding the product $2i \cdot 2$ with the product $2 \cdot -3i$, which gives $4i - 6i = -2i$. On the right-hand side, however, the imaginary part is $2i + 4i = 6i$, so the equation is false. The two complex numbers don't have equal imaginary parts, so they can't be equal.)

Then, display this equation for all to see:

$$(3 - 5i)(-2 + 3i) = (-10 + 6i) + (4 + 13i)$$

Again, ask students to use arguments about the structure of the expressions on each side of the equation to mentally determine whether or not it is true. (The imaginary parts match because on the left-hand side, $3 \cdot 3i + -5i \cdot -2 = 9i + 10i = 19i$, and on the right-hand side, $6i + 13i = 19i$. The real parts don't match, however. On the right-hand side, the real part is $-10 + 4 = -6$. On the left-hand side, the real parts of the two factors also multiply to -6, but there are multiples of i^2 that are real that haven't been accounted for.)

If time allows, ask students to verify the arguments for one or both equations by writing each side in the form $a + bi$, where a and b are real numbers.

10.4 Squares and Imaginary Numbers

Cool-down

5 min

Standards

Addressing HSN-CN.A.1

Student Task Statement

1. Write $-(7i)^2$ as either an integer or an integer multiple of i.
2. Multiply these complex numbers. Write the solution in the form $a + bi$, where a and b are real numbers. $(2 + 5i)(4 - i)$

Student Response

1. 49
2. $13 + 18i$

Responding to Student Thinking

Points to Emphasize
If students struggle to multiply, add, or subtract complex numbers, use the optional lesson referred to here to provide additional practice.

Integrated Math 2, Unit 6, Lesson 11 More Arithmetic with Complex Numbers

Lesson 10 Summary

One way to multiply two complex numbers is to use the distributive property.

$$(2 + 3i)(4 + 5i) = 8 + 10i + 12i + 15i^2$$

Remember that $i^2 = -1$, so:

$$(2 + 3i)(4 + 5i) = 8 + 10i + 12i - 15$$

When we add the real parts together and the imaginary parts together, we get:

$$(2 + 3i)(4 + 5i) = -7 + 22i$$

Lesson 10 Practice Problems

1 Student Task Statement

Which expression is equivalent to $2i(5+3i)$?

A. $-6+10i$

B. $6+10i$

C. $-10+6i$

D. $10+6i$

Solution

A

2 Student Task Statement

Lin says, "When you add or multiply two complex numbers, you will always get an answer you can write in $a+bi$ form."

Noah says, "I don't think so. Here are some exceptions I found:"

$(7+2i)+(3-2i)=10$

$(2+2i)(2+2i)=8i$

a. Check Noah's arithmetic. Is it correct?

b. Can Noah's answers be written in the form $a+bi$, where a and b are real numbers? Explain or show your reasoning.

Solution

a. Yes.

b. Both can be written in $a+bi$ form. $10=10+0i$, and $8i=0+8i$.

3 Student Task Statement

Explain to someone who missed class how you would write $(3-5i)(-2+4i)$ in the form $a+bi$, where a and b are real numbers.

Solution

Sample response: You could distribute each part of one expression to the other one and then collect like terms. You would get $-6+12i+10i-20i^2$. Because $i^2=-1$, $-20i^2=20$. So the final result is $14+22i$.

4 from Unit 6, Lesson 4

Student Task Statement

Which expression is equal to $729^{\frac{2}{3}}$?

A. 243

B. 486

C. 9^2

D. 27^3

Solution

C

5 from Unit 3, Lesson 8

Student Task Statement

Find the real solution(s) to each equation, or explain why there is no real solution.

a. $2x^2 - \frac{2}{3} = 5\frac{1}{3}$

b. $(x+1)^2 = 81$

c. $3x^2 + 14 = 12$

Solution

a. $x = \sqrt{3}$ or $-\sqrt{3}$

b. $x = 8$ or -10

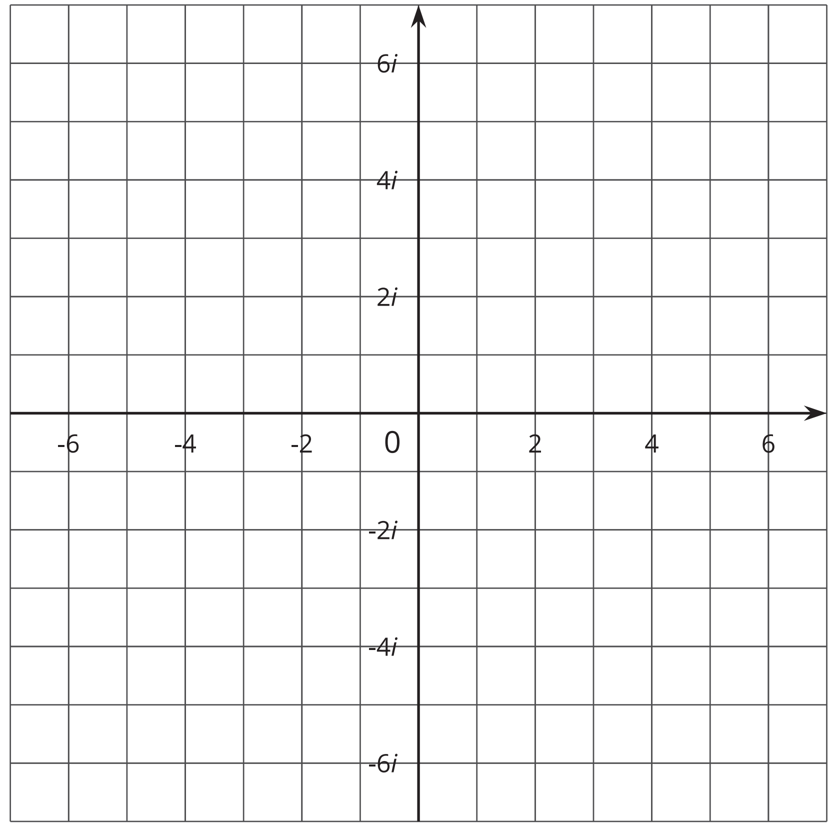
c. No real solutions. After subtracting 14 from each side, the equation is $3x^2 = -2$. The left side is positive and the right side is negative.

6 from Unit 6, Lesson 8

Student Task Statement

Plot each number in the complex plane.

a. $5i$
b. $2+4i$
c. -3
d. $1-3i$
e. $-5-2i$



Solution