Unit 1, Lesson 9

Speedy Delivery

Goals

- Choose geometric methods to solve design problems.
- Construct perpendicular bisectors and explain (in writing) how they are used to solve problems.

Learning Targets

- I can construct perpendicular bisectors to help solve problems.
- I can use my geometry knowledge to solve problems.

Lesson Narrative

In this lesson, students build on their experiences with perpendicular bisectors by creating Voronoi (vo-ro-NOY) diagrams to answer questions about allocating resources in a real-world situation (MP4). There are two optional activities that both provide the opportunity to apply another Voronoi diagram. One focuses on additional aspects of the mathematical modeling cycle, while the other introduces tessellations.

Some of the activities in this lesson work best when each student has access to the GeoGebra Geometry Tool from Math Tools or at https://im612.org/geometry-tool. Using the GeoGebra Geometry Tool rather than the GeoGebra Constructions Tool gives students access to a perpendicular bisector tool.

Math Community

Today's community building centers on the teacher sharing their draft commitments as part of the mathematical community. At the end of the lesson, students are invited to suggest additions to the teacher sections of the chart.

Standards

Building On	**HSG-CO.D.12**
Addressing	**HSG-CO.D.12, HSG-MG.A.3, HSN-Q.A.2, HSN-Q.A.3**

Instructional Routines

- Aspects of Mathematical Modeling
- Draw It
- MLR6: Three Reads
- MLR8: Discussion Supports

Required Materials

Materials to Gather

- Math Community Chart: Activity 1, Cool-down
- Dynamic geometry software: Activity 2, Activity 3

Materials to Copy

- Another Layer Handout (1 copy for every 1 students): Activity 4

Required Preparation

Activity 1:

In the "Doing Math" teacher section of the Math Community Chart, add 2–5 commitments you have for what your

teaching practice “looks like” and “sounds like” this year.

Activity 2:

Acquire computers or tablets that can run the GeoGebra Geometry Tool from Math Tools (https://im612.org/geometry-tool) or similar dynamic geometry software, with one for every 2–3 students. The digital version is recommended for all classes over the paper and pencil version.

Ensure that students have at least 4 colors in their geometry toolkits if they will be doing the paper and pencil version of “Who Is Closest?”

Activity 3:

Acquire computers or tablets that can run GeoGebra Geometry from Math Tools, with one for every 2–3 students. The digital version is recommended for all classes over the paper and pencil version.

Activity 4:

Acquire computers or tablets that can run GeoGebra Geometry from Math Tools, with one for every 2–3 students. The digital version is recommended for all classes over the paper and pencil version.

Student Facing Learning Goals

Let’s use perpendicular bisectors.

9.1 Divide the Map

5 min

Warm-up

Activity Narrative

In this activity, students are invited to make a choice without explicit instructions on how to divide the map. Monitor for students who use a perpendicular bisector and students who take the context into consideration.

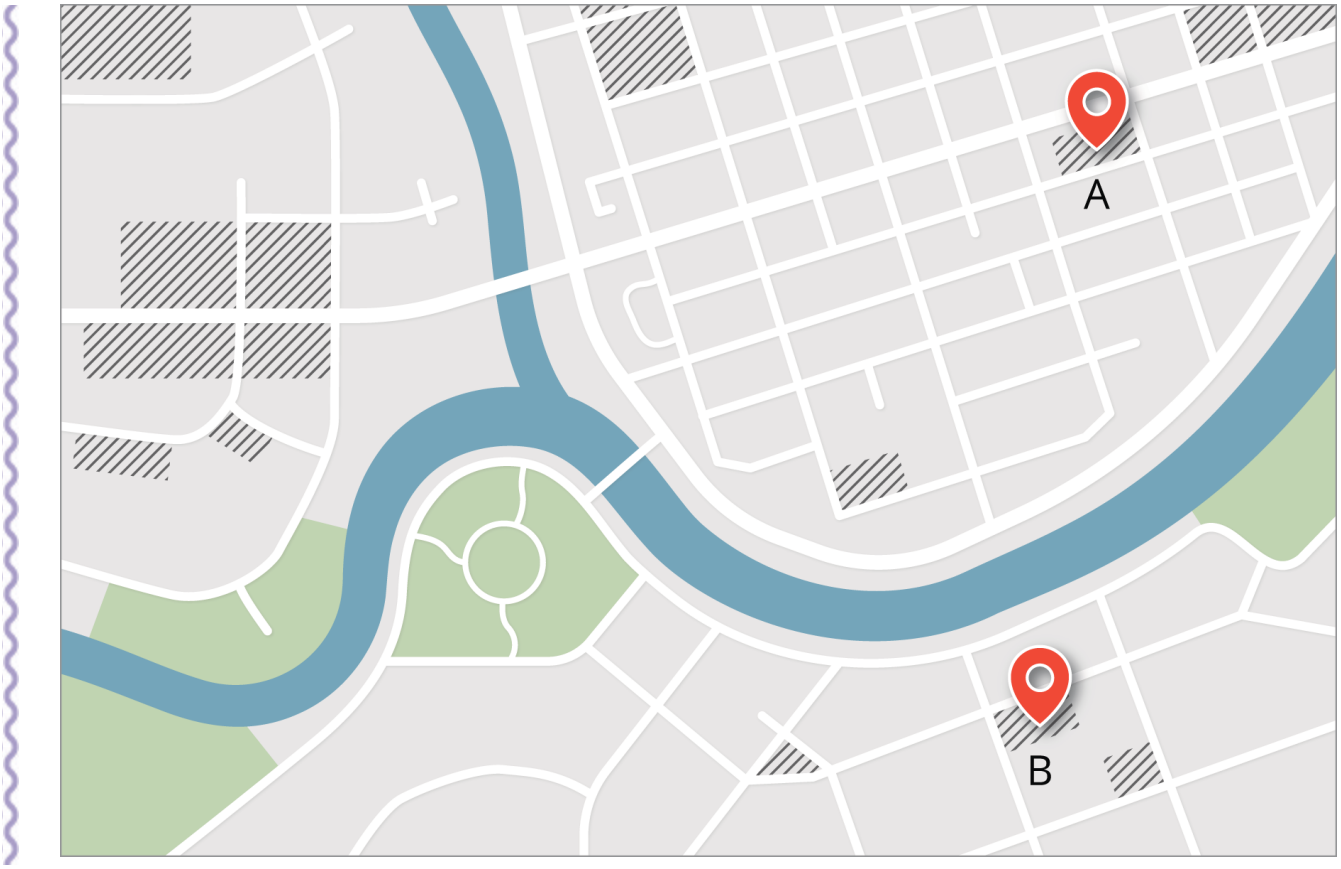
Students will examine a simplified version of this diagram in the next activity and then have the opportunity to interact with an even more complex version in the optional activity “Now Who Is Closest?”

Standards

Addressing **HSG-MG.A.3**

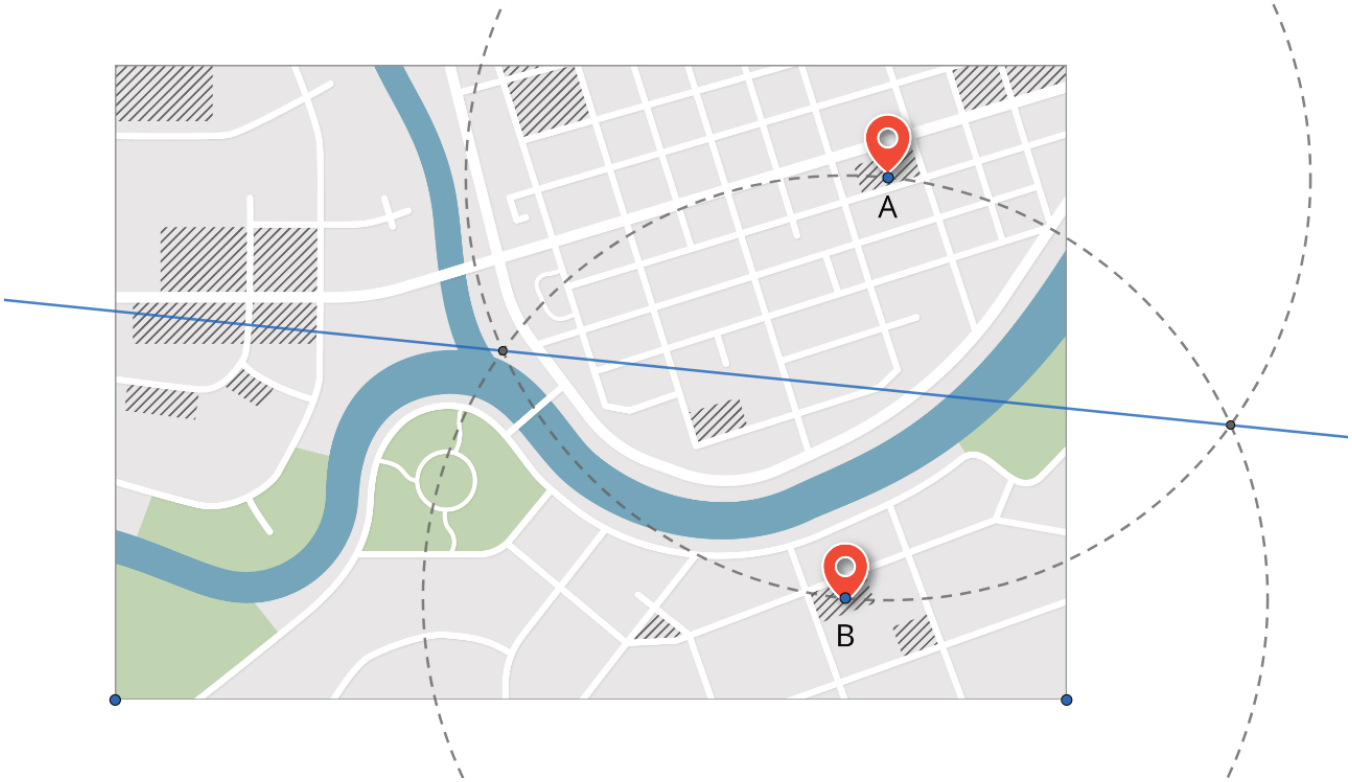
Student Task Statement

A company has two store locations. They offer delivery services for their product. Divide the map into two regions so the computer system knows which store to direct a delivery order to.



Student Response

Sample response:



Activity Synthesis

Invite students to share their regions and their reasoning. Acknowledge that there isn't enough information to make a final decision, so there is no one right answer. Explain that the shaded areas on the map represent large buildings.

Math Community

After the *Warm-up*, display the Math Community Chart with the "Doing Math" actions added to the teacher section for all to see. Give students 1 minute to review. Then share 2–3 key points from the teacher section and your reasoning for adding them. For example,

- If "questioning vs. telling," a shared reason could focus on your belief that students are capable mathematical thinkers and your desire to understand how students are making meaning of the mathematics.
- If "listening," a shared reason could be that sometimes you want to sit quietly with a group just to listen and hear student thinking and not because you think the group needs help or is off-track.

After sharing, tell students that they will have the opportunity to suggest additions to the teacher section during the *Cool-down*.

9.2 Who Is Closest?

15 min

Activity Narrative

There is a digital version of this activity.

In this activity, students are building skills that will help them in mathematical modeling (MP4). The method of modeling stores in a city with points in a square is provided, but students need to realize they can use perpendicular bisectors to determine which stores should take responsibility for which parts of the city. Students also need to decide how to allocate the 100 employees. It is expected that students who do not use digital tools to measure areas will approximate areas using decomposition techniques and estimation. Students working on paper will likely not have time for the fourth store analysis. These students do not need to see this question to participate in the discussion.

This activity works best when each student has access to the GeoGebra Geometry Tool from Math Tools (https://im612.org/geometry-tool) or similar dynamic geometry software because it would take too long to do otherwise. If students don't have individual access, projecting the GeoGebra Geometry Tool would be helpful during the *Activity Synthesis*.

This is the first time Math Language Routine 6: *Three Reads* is suggested in this course. In this routine, students are supported in reading a mathematical text, situation, or word problem three times, each with a particular focus. During the first read, students focus on comprehending the situation. During the second read, students identify quantities. During the third read, the final prompt is revealed and students brainstorm possible starting points for answering the question. The intended question is withheld until the third read so students can make sense of the whole context before rushing down a solution path. The purpose of this routine is to support students' reading comprehension as they make sense of mathematical situations and information through conversation with a partner.

Access for English Language Learners

This activity uses the *Three Reads* math language routine to advance reading and representing as students make sense of what is happening in the text.

Standards

Addressing **HSG-CO.D.12, HSG-MG.A.3, HSN-Q.A.2, HSN-Q.A.3**

Instructional Routines

- Aspects of Mathematical Modeling
- Draw It
- MLR6: Three Reads

Launch

Arrange students in groups of 2. Use *Three Reads* to support reading comprehension and sense-making about this problem. Display only the problem stem and image, without revealing the questions.

- In the first read, students read the problem with the goal of comprehending the situation.
  For the first read, read the problem aloud while everyone else reads along, and then ask, "What is this situation about? What is going on here?" Allow 1 minute to discuss with a partner and then share with the whole class. A typical response may be, "There are three stores of the same type in a city located at each of the labeled points on the map." Listen for and clarify any questions about the context.
- In the second read, students analyze the mathematical structure of the story by naming quantities.
  Invite students to read the problem aloud with their partner, or select a student to read to the class, then prompt students by asking, "What can be counted or measured in this situation?" Give students 30 seconds of quiet think time, followed by another 30 seconds to share with their partner. A typical response may be: "Points E, F, and G represent the locations of the same store. I can measure distances between the points or angles made by connecting some points."
- In the third read, students brainstorm possible starting points for answering the questions.
  Invite students to read the problem aloud with their partner, or select a different student to read to the class. After the third read, reveal the first question on breaking the city down into regions and ask, "What are some ways we

might get started on this?" Instruct students to think of ways to approach the questions without actually solving. Give students 1 minute of quiet think time followed by another minute to discuss with their partner. Invite students to name some possible strategies referencing quantities from the second read. Provide these sentence frames as partners discuss: "To represent some points that are closest to Store E, I can . . . ." and "Points that are equally far from Store E and Store F can be constructed using . . . ."
As partners are discussing their solution strategies, select 1–2 students to share their ideas with the whole class. As students are presenting their strategies to the whole class, create a display that summarizes starting points for each question. (Stop students as needed before they share complete solutions or answers.)

Give students time to complete the rest of the activity followed by a whole-class discussion.

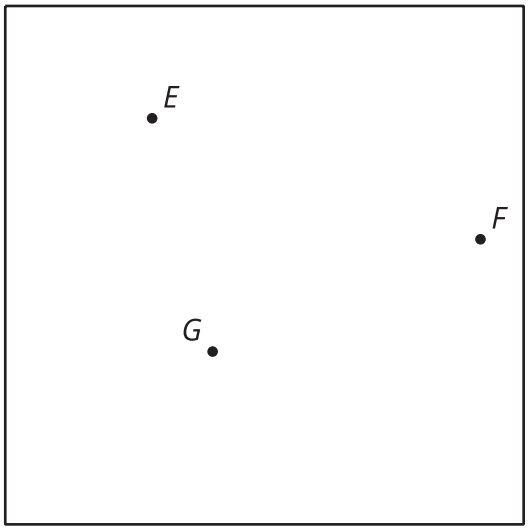
If it does not come up during the third read, invite students to use different colors to begin shading the square. "Choose 1 color for point E. Shade any spot you know is closer to E than point F or point G. Repeat using a new color for each of the other 2 points."

Access for Students with Disabilities

Representation: Access for Perception. Use physical objects such as coins for the stores and pencils as the dividing lines to demonstrate the context.
Supports accessibility for: Conceptual Processing, Language, Memory

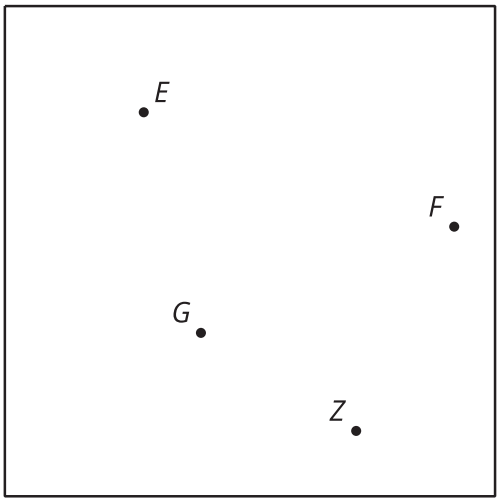
Student Task Statement

Here is a map of a square city with 3 locations of the same store.



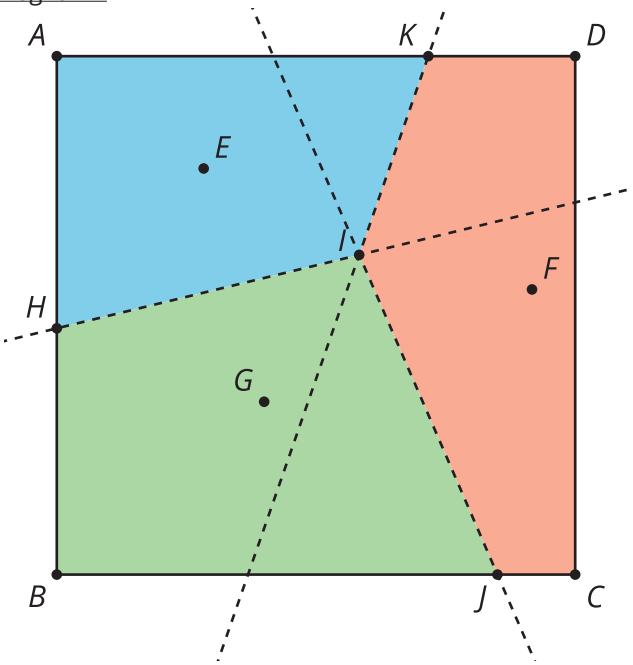
1. The company wants to break the city down into regions so that whenever someone orders from an address, their order is delivered from the store closest to them. They have hired you to decide how to partition the city between the 3 stores. Explain or show your reasoning.
2. If there are 100 employees, how should they be distributed among the 3 locations?
3. Is there anywhere in the city that has the same distance to all 3 stores?

4. Now a fourth store opens. Partition the city again.



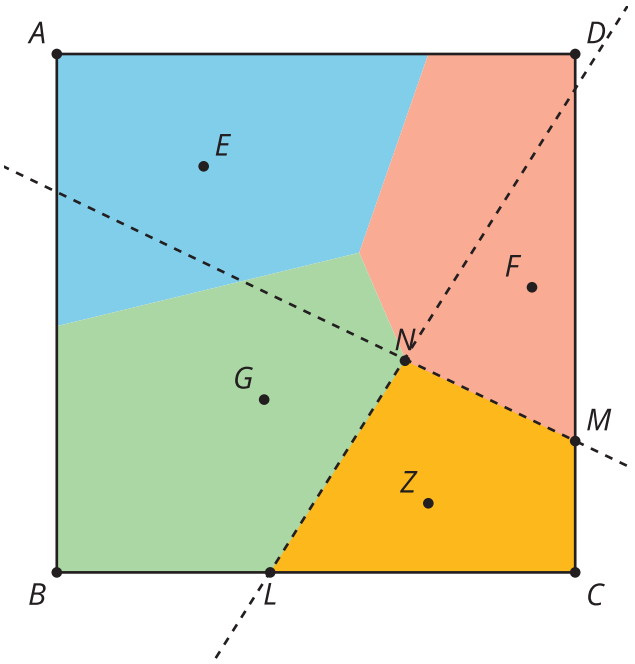
Student Response

1. https://www.geogebra.org/m/k8gk6fvz



Sample reasoning: I drew the perpendicular bisectors between each pair of stores to partition the city.

2. The area around point E is roughly 28% of the total area, so approximately 28 workers should be at that location. The area around point G is roughly 40% of the total area, so approximately 40 workers should be at that location. That leaves the store at point F with approximately 32 workers.
3. Yes, the intersection of the perpendicular bisectors is the same distance to all 3 stores.



4.

Building on Student Thinking

If a student is struggling to start, ask them to consider what would happen if there were only 2 stores. Ask them to think about their experience from the construction techniques lessons and if they can figure out a way to separate the points closer to one of the two stores from the points that are closer to the other store.

If a student is stuck finding the area on paper, either encourage them to break the shapes into simpler pieces or to estimate, depending on time. If a student is stuck finding the area on the applet, show them the area tool under the measurement menu (look for the angle icon).

Are You Ready for More?

In 1854 there was an outbreak of cholera in London. A physician named John Snow thought the water supply might be responsible. He made a map showing the location of all the water pumps in the city and the locations of all the deaths due to cholera in the city. How could he have used the ideas in this activity to help isolate the cause of the outbreak?

The diagrams you made in the activity and that Snow made are called Voronoi diagrams, and they are still actively studied by mathematicians.

Extension Student Response

Sample response: He could have divided the city up by which regions were closest to which water pumps. He then might have observed that almost all the deaths were in the region closest to one particular water pump.

Activity Synthesis

The purpose of this discussion is to highlight students' level of confidence in the accuracy of the model.

Here are some questions for discussion:

- "How did you decide to divide the city?" (I used perpendicular bisectors to create regions that were closest to each point.)
- "How did you decide how many employees to station at each store?" (I found the area of each region and split the 100 people accordingly.)
- "How confident are you that this is the perfect strategy?" (Not very, I don't have any other information, such as how many orders come from each region on average.)

If not brought up by students, ask whether it would be appropriate to assign 28.2 workers to a location. (No, two-tenths of a person doesn't make sense. Yes, if someone spent part of their week at one location and part at a different location.)

Tell students that without additional information, a geometry interpretation is a good starting point. This technique for partitioning using perpendicular bisectors is called a Voronoi diagram. It is named for Georgy (JOHR-jee) Voronoy, a nineteenth-century Ukranian mathematician who defined this process.

9.3 Now Who Is Closest?

15 min

Optional

Activity Narrative

In this optional activity, students continue to build skills that will help them in mathematical modeling (MP4). Students can either continue studying the same map provided or use a map of their choosing. Instead of creating regions for store deliveries, students could study a map of school districts, grocery stores, or polling places.

Suggest students use different colors or markings to indicate the regions for the first and last parts of the activity.

Making dynamic geometry software available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Addressing **HSG-CO.D.12, HSG-MG.A.3**

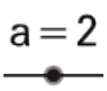
Instructional Routines

- Aspects of Mathematical Modeling
- Draw It
- MLR8: Discussion Supports

Launch

If students will be using a map of their choosing, invite students to think of other situations where a Voronoi diagram could be useful. Instruct them to find a map to use for the activity.

If students will be using a digital tool to construct their diagrams, demonstrate how to save an image to their devices, then import an image into geogebra.org/classic by using the Image Tool, located in the drop-down menu with the slider icon.

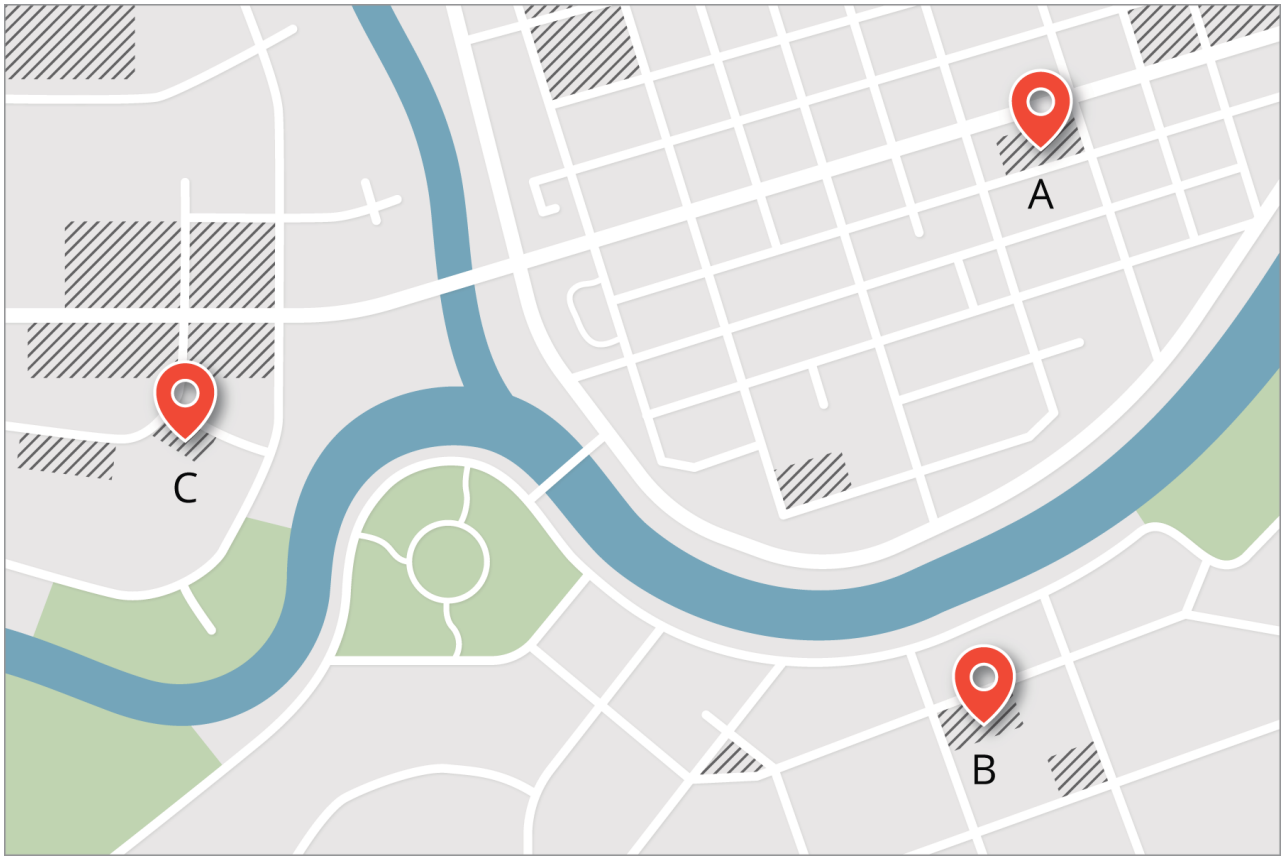


Access for English Language Learners

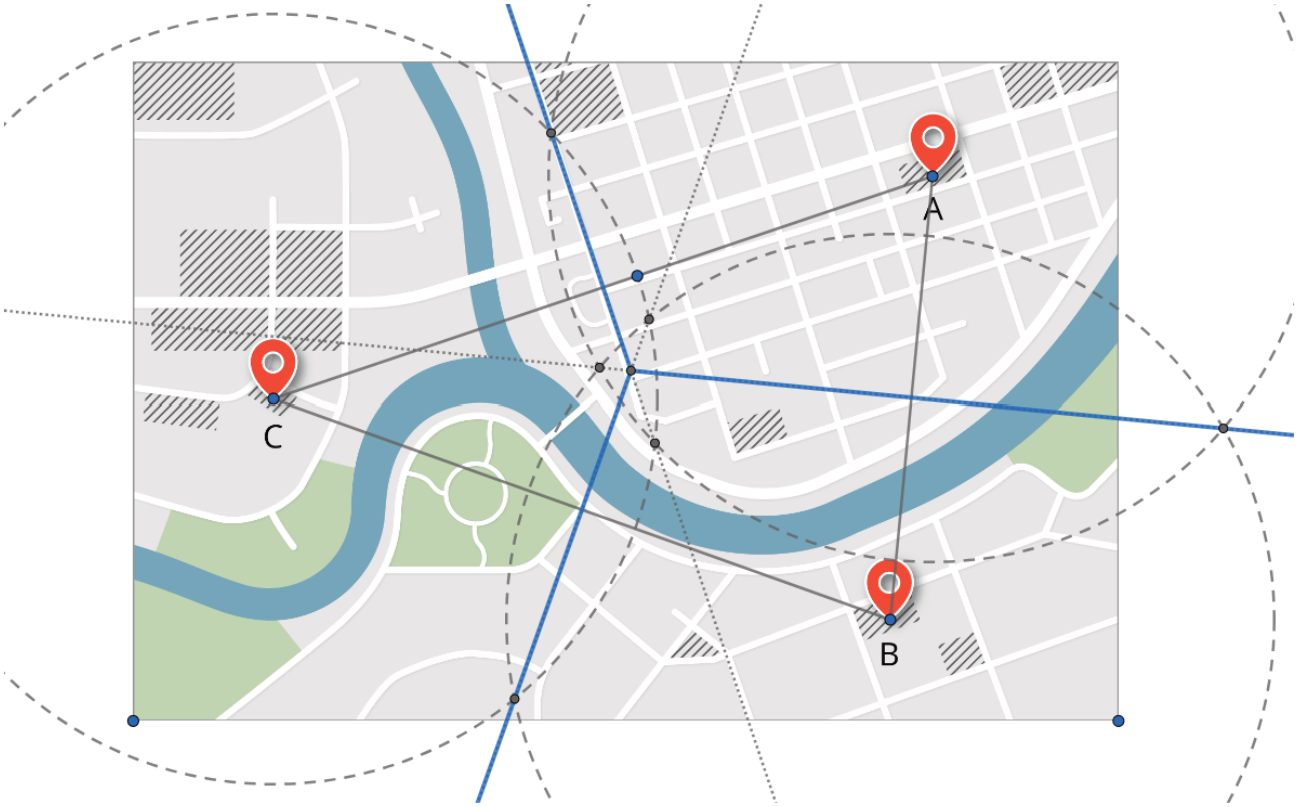
MLR8 Discussion Supports. Students who are working toward verbal output may benefit from access to mini whiteboards, sticky notes, or spare paper to write down and show their responses to their partner.
Advances: Writing, Representing

Student Task Statement

1. Create a Voronoi diagram to show the regions that are closest to each store. Consider only the points, without any of the surrounding information.
2. Look at each of the regions. What do you notice? What do you wonder?
3. Partition the map again. This time consider the points and the surrounding information when creating the regions.



Student Response

1. 

2. Sample response: I notice that the region with C has a lot more large buildings than the other regions. I wonder what the area just outside of the map looks like.
3. Sample response: It will take much longer to get from C to the area south of the river if they have to cross 2 bridges. There may be more population in the area around C since it has more large buildings.

Activity Synthesis

Ask students to share their responses and display their responses for all to see.

Ask students:

- "How is this activity the same as the previous activity?" (We are still dividing the map into regions closest to the given points of interest.)
- "How is it different?" (There is more information to consider.)

9.4 Another Layer

Optional

15 min

Activity Narrative

There is a digital version of this activity.

This optional activity builds on the "Who Is Closest?" activity by employing the same technique of finding regions that are closest to certain points and decorating those regions to make new and interesting patterns.

Making dynamic geometry software available gives students an opportunity to choose appropriate tools strategically (MP5).

Standards

Building On **HSG-CO.D.12**

Instructional Routines

- Draw It

Launch

Give each student a tessellation from the blackline master. Tell students that a tessellation is a regular repeating pattern of one or more shapes that cover the entire plane. Tell students that to save time, it is OK for them to use paper folding or make estimates rather than use formal straightedge and compass construction techniques to make perpendicular bisectors.

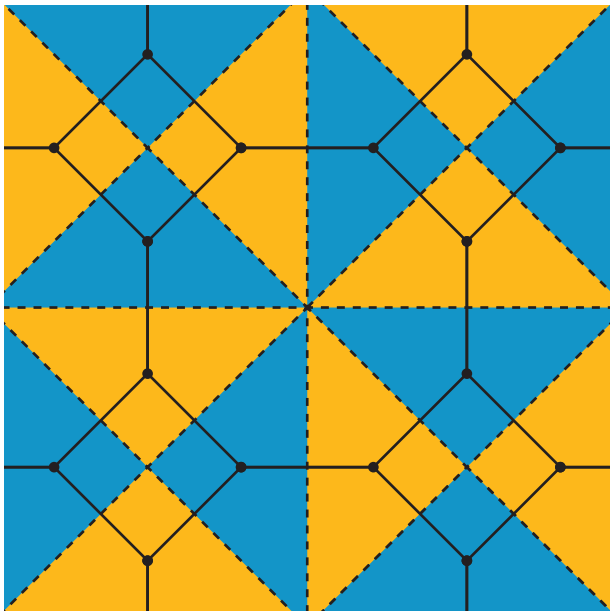
Student Task Statement

Your teacher will give you a tessellation.

1. Mark the intersection points on the tessellation.
2. Imagine that each point is a store from the "Who Is Closest?" activity. Repeat the process you used there to define the regions that are closest to each of the points.
3. Use color or shading to enhance your design.

Student Response

Sample response: (from the tessellation of octagons and squares)



Activity Synthesis

Display student responses for all to see. Invite students to discuss if their new diagram is a tessellation.

Lesson Synthesis

Display the images of store locations from the *Warm-up* and activities.

Ask students to brainstorm a list of additional factors that would impact how the regions are divided and how many employees work in each location. (population density in each region, whether the store provides supplies to businesses or residential neighborhoods, traffic and road accessibility)

Invite students to discuss with a partner the topics they would discuss in a meeting with the owner of the company.

9.5 Write a Letter

Cool-down

5 min

Standards

Addressing **HSG-MG.A.3**

Launch

If students engaged in the optional activity with maps of their choosing, tell them they can write a letter to the relevant stakeholder instead of the owner of the company from the earlier activity.

Math Community

Before distributing the *Cool-downs*, display the Math Community Chart and the community building question "What additions would you make to the teacher 'Doing Math' section of the Math Community Chart?" Ask students to respond to the question after completing the *Cool-down* on the same sheet.

After collecting the *Cool-downs*, identify themes from the community building question. Use them to add to or revise the teacher "Doing Math" section of the Math Community Chart before Exercise 4.

Student Task Statement

Choose one of the diagrams you drew today of the regions around stores. Write a letter to the owner of the company explaining what the diagram you drew tells them. Ask for any information you need to decide how to distribute the employees among the stores.

Student Response

Sample response:

Dear company owner,

The map of the city shows you which store is closest to each part of the city. When an order comes in, find the address on the map. There will be a store in the same region as the address, without the delivery person crossing any segments. If the address is on an edge, then two stores are the same distance away, and either one can deliver it.

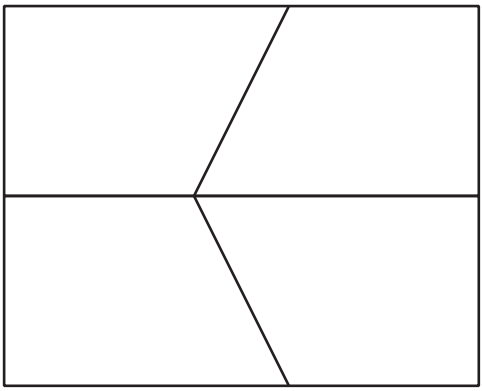
To know best how to distribute the 100 employees, I need information on which region has the largest number of customers. The more orders that come from a certain section of the map, the more people that should be working in the store in that section.

Responding to Student Thinking

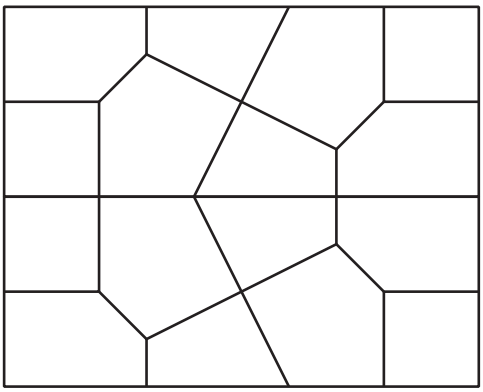
More Chances
Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Lesson 9 Summary

A tessellation is an arrangement of figures that covers the entire plane without gaps or overlaps. A simple example is a square grid. So that means graph paper is a tessellation. Here is another tessellation made of quadrilaterals. Can you see how repeating this pattern could cover the entire plane?



One way to draw a new tessellation is to decompose the plane into regions that are closest to each vertex. This method uses perpendicular bisectors and is called a Voronoi diagram. It is also a tessellation. What would this pattern look like when it is extended to cover the entire plane?



Lesson 9 Practice Problems

1 Student Task Statement

Which construction can be used to determine whether point C is closer to point A or point B?

A. Construct triangle ABC.

B. Construct a line perpendicular to segment AB through point C.

C. Construct the bisector of angle ACB.

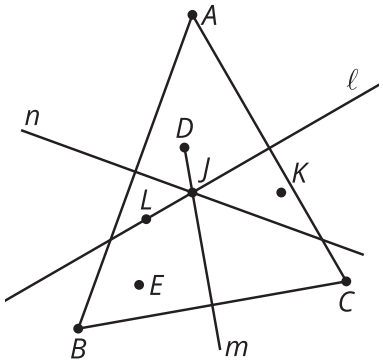
D. Construct the perpendicular bisector of segment AB.

Solution

D

2 Student Task Statement

The diagram is a straightedge and compass construction. Lines ℓ, m, and n are the perpendicular bisectors of the sides of triangle ABC. Select **all** the true statements.



A. Point E is closer to point A than it is to point C.

B. Point L is closer to point B than it is to point A.

C. Point D is closer to point B than it is to point C.

D. Point J is closer to point A than it is to point B or point C.

E. Point K is closer to point C than it is to point A or point B.

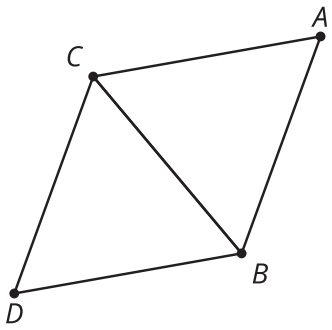
F. Point L is closer to point C than it is to point A or point B.

Solution

B, E

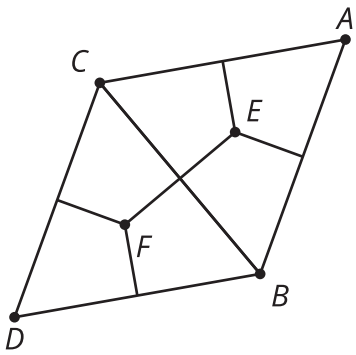
3 Student Task Statement

Decompose the figure into regions that are closest to each vertex. Explain or show your reasoning.



Solution

Sample response: To divide the triangles into these regions, construct the perpendicular bisector of each segment. The perpendicular bisectors intersect and divide each triangle into three regions. The points in each region are those closest to the vertex in that region.



4 Student Task Statement

Which construction could be used to construct an isosceles triangle ABC given line segment AB?

A. Mark a third point C that is not on segment AB. Draw segments AC and BC.

B. Label a point C on segment AB and construct a line perpendicular to AB through point C. Draw segments AC and BC.

C. Construct the perpendicular bisector of segment AB. Mark the intersection of this line and AB and label it C. Draw segments AC and BC.

D. Construct the perpendicular bisector of segment AB. Mark any point C on the perpendicular bisector except where it intersects AB. Draw segments AC and BC.

Solution

D

5 from Unit 1, Lesson 7

Student Task Statement

Select **all** true statements about regular polygons.

A. All angles are right angles.

B. All angles are congruent.

C. All side lengths are equal.

D. There are exactly 4 sides.

E. There are at least 3 sides.

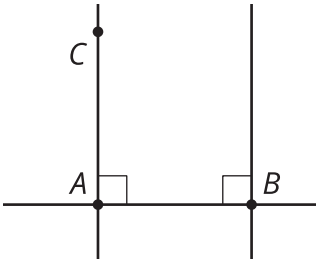
Solution

B, C, E

6 from Unit 1, Lesson 7

Student Task Statement

This diagram shows the beginning of a straightedge and compass construction of a rectangle.



The construction followed these steps:

a. Start with two marked points A and B.

b. Use a straightedge to construct line AB.

c. Use a previous construction to construct a line perpendicular to AB passing through A.

d. Use a previous construction to construct a line perpendicular to AB passing through B.

e. Mark a point C on the line perpendicular to AB passing through A.

Explain the steps needed to complete this construction.

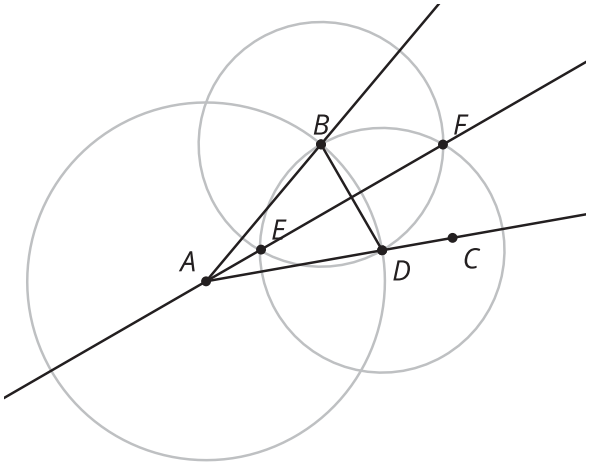
Solution

Sample response: Construct a line perpendicular to AC passing through point C. Mark the intersection point of this line and the line perpendicular to AB passing through B, and label it D. Draw segments AC, CD, and DB.

7 from Unit 1, Lesson 5

Student Task Statement

This diagram is a straightedge and compass construction of an angle bisector. Is it important that the circle with center B passes through D and that the circle with center D passes through B? Show or explain your reasoning.



Solution

Sample response: No, I could change the radius of these circles so the one with center B does not contain D and the one with center D does not contain B. It is important that these circles have the same radius. It is also important that the radius be big enough that the two circles meet in a pair of points.