Using Quadratic Expressions in Vertex Form to Solve Problems

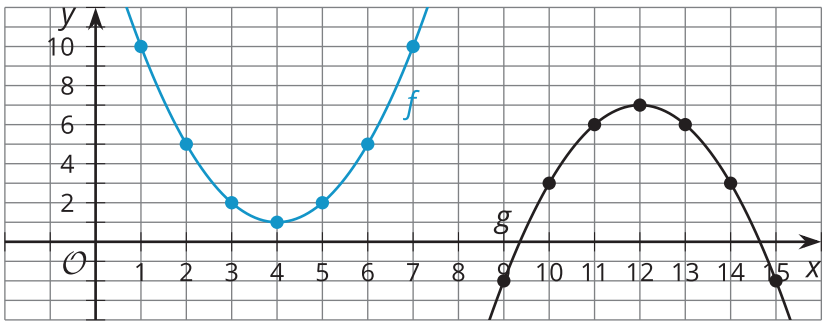
Let's find the maximum or minimum value of a quadratic function.

23.1 Values of a Function

Here are graphs that represent two functions, f and g, defined by these equations:

$$f(x) = (x-4)^2 + 1$$

$$g(x) = -(x-12)^2 + 7$$



1. $f(1)$ can be expressed in words as "the value of f when x is 1." Find or compute:
   a. the value of f when x is 1
   b. $f(3)$
   c. $f(10)$
2. Does f have a maximum, minimum, or neither? If it has a maximum or minimum, what is the greatest or least value $f(x)$ can have?

3. $g(9)$ can be expressed in words as "the value of g when x is 9." Find or compute:
   a. the value of g when x is 9
   b. $g(13)$
   c. $g(2)$
4. Does g have a maximum, minimum, or neither? If it has a maximum or minimum, what is the greatest or least value $g(x)$ can have?

23.2 Maximums and Minimums

1. The graph that represents $p(x) = (x - 8)^2 + 1$ has its vertex at $(8, 1)$. Here is one way to show, without graphing, that $(8, 1)$ corresponds to the *minimum* value of p.

   - When $x = 8$, the value of $(x - 8)^2$ is 0, because $(8 - 8)^2 = 0^2 = 0$.
   - Squaring any nonzero number always results in a positive number, so when x is any value other than 8, $(x - 8)$ will be a number other than 0, and when the expression is squared, $(x - 8)^2$, it will be positive.
   - Any positive number is greater than 0, so when $x \neq 8$, the value of $(x - 8)^2$ will be greater than when $x = 8$. In other words, p has the least value when $x = 8$.

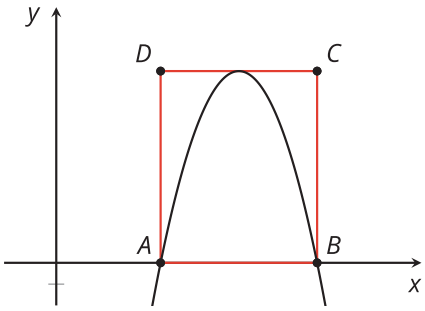
   Use similar reasoning to explain why the point $(4, 1)$ corresponds to the *maximum* value of q, defined by $q(x) = -2(x - 4)^2 + 1$.

2. Here are some quadratic functions and the coordinates of the vertex of the graph of each. Determine if the vertex corresponds to the maximum or the minimum value of the function. Be prepared to explain how you know.

equation	coordinates of the vertex	maximum or minimum?
$f(x) = -(x - 4)^2 + 6$	$(4, 6)$	
$g(x) = (x + 7)^2 - 3$	$(-7, -3)$	
$h(x) = 4(x + 5)^2 + 7$	$(-5, 7)$	
$k(x) = x^2 - 6x - 3$	$(3, -12)$	
$m(x) = -x^2 + 8x$	$(4, 16)$	

Are you ready for more?

Here is a portion of the graph of function q, defined by $q(x) = -x^2 + 14x - 40$.



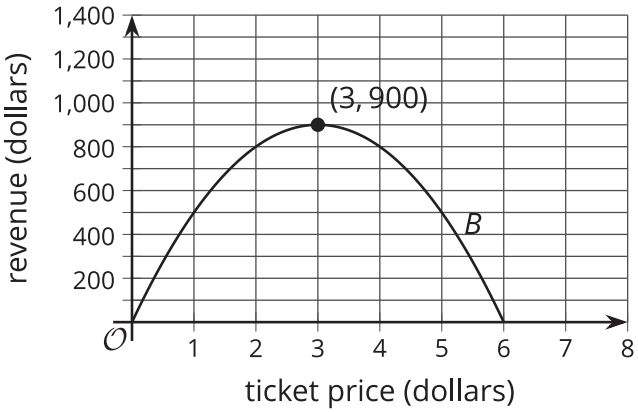
$ABCD$ is a rectangle. Points A and B coincide with the x-intercepts of the graph, and segment CD just touches the vertex of the graph.

Find the area of $ABCD$. Show your reasoning.

23.3 All the World's a Stage

Function A, defined by $p(600 - 75p)$, describes the revenue collected from the sales of tickets for Performance A, a musical.

The graph represents a function, B, that models the revenue collected from the sales of tickets for Performance B, a Shakespearean comedy.



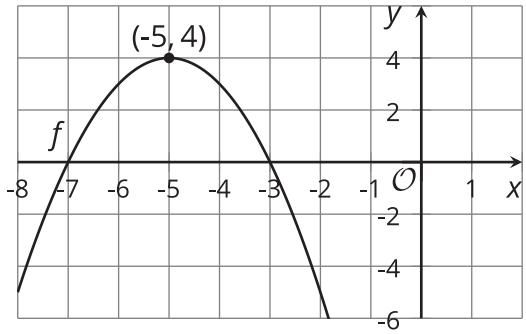
In both functions, p represents the price of one ticket, and both revenues and prices are measured in dollars.

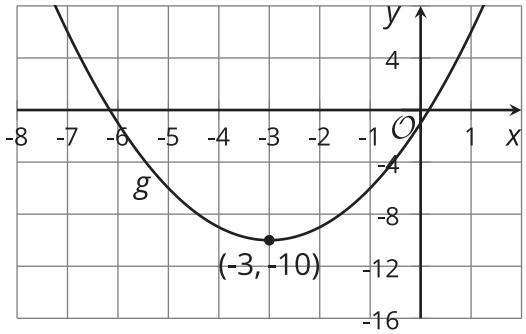
Without creating a graph of A, determine which performance gives the greater maximum revenue when tickets are p dollars each. Explain or show your reasoning.

Lesson 23 Summary

Any quadratic function has either a *maximum* or a *minimum* value. We can tell whether a quadratic function has a maximum or a minimum by observing the vertex of its graph.

Here are graphs representing functions f and g, defined by $f(x) = -(x+5)^2 + 4$ and $g(x) = x^2 + 6x - 1$.





- The vertex of the graph of f is $(-5, 4)$, and the graph is a parabola that opens downward.
- No other points on the graph of f (no matter how much we zoom out) are higher than $(-5, 4)$, so we can say that f has a maximum of 4, and that this occurs when $x = -5$.

- The vertex of the graph of g is at $(-3, -10)$, and the graph is a parabola that opens upward.
- No other points on the graph (no matter how much we zoom out) are lower than $(-3, -10)$, so we can say that g has a minimum of -10, and that this occurs when $x = -3$.

We know that a quadratic expression in vertex form can reveal the vertex of the graph, so we don't actually have to graph the expression. But how do we know, without graphing, if the vertex corresponds to a maximum or a minimum value of a function?

The vertex form can give us that information as well!

To see if $(-3, -10)$ is a minimum or maximum of g, we can rewrite $x^2 + 6x - 1$ in vertex form, which is $(x+3)^2 - 10$. Let's look at the squared term in $(x+3)^2 - 10$.

- When $x = -3$, $(x+3)$ is 0, so $(x+3)^2$ is also 0.
- When x is not -3, the expression $(x+3)$ is a nonzero number, and $(x+3)^2$ is positive.
- Because a squared number cannot have a value less than 0, $(x+3)^2$ has the least value when $x = -3$.

To see if $(-5, 4)$ is a minimum or maximum of f, let's look at the squared term in $-(x+5)^2 + 4$.

- When $x = -5$, $(x+5)$ is 0, so $(x+5)^2$ is also 0.
- When x is not -5, the expression $(x+5)$ is nonzero, so $(x+5)^2$ is positive. The expression $-(x+5)^2$ has a coefficient of -1, however. Multiplying $(x+5)^2$ (which is positive when $x \neq -5$) by a negative number results in a negative number.
- Because a negative number is always less than 0, the value of $-(x+5)^2 + 4$ will always be less when $x \neq -5$ than when $x = -5$. This means $x = -5$ gives the greatest value of f.