Unit 4, Lesson 5

Changes Over Rational Intervals

Let's look at how an exponential function changes when the input changes by a fractional amount.

5.1 Changes Over Intervals

Consider the exponential function $h(x) = 4^x$. For each question, be prepared to share your reasoning with the class.

1. By what factor does h increase when the exponent x increases by 1?

2. By what factor does h increase when the exponent x increases by 2?

3. By what factor does h increase when the exponent x increases by 0.5?

5.2 Machine Depreciation

After purchase, the value of a machine depreciates exponentially. The table shows its value as a function of years since purchase.

years since purchase	value in dollars
0	16,000
0.5	
1	13,600
1.5	
2	11,560
3	9,826

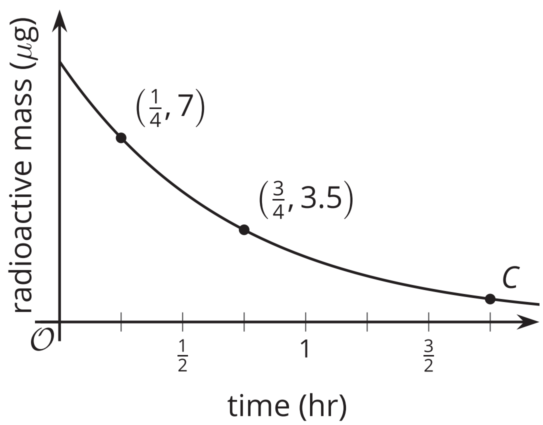
1. The value of the machine in dollars is a function f of time t, the number of years since the machine was purchased. Find an equation defining f, and be prepared to explain your reasoning.
2. Find the value of the machine when t is 0.5 and 1.5. Record the values in the table.
3. Observe the values in the table. By what factor did the value of the machine change:
   a. every one year, say from 1 year to 2 years, or from 0.5 year to 1.5 years?
   b. every half a year, say from 0 years to 0.5 year, or from 1.5 years to 2 years?
4. Suppose we know $f(q)$, the value of the machine q years since purchase. Explain how we could use $f(q)$ to find $f(q+0.5)$, the value of the machine half a year after that point.

Are you ready for more?

A bank account is growing exponentially. At the beginning of 2010, the balance, in dollars, was 1,200. At the beginning of 2015, the balance was 1,350. What is the bank account balance at the beginning of 2018? Give an approximate answer as well as an exact expression.

5.3 Tracing the Radioactivity

A small leak occurs in a radioactive containment vessel. The leak is detected 15 minutes after the leak begins. The amount of radioactive material is measured, in micrograms (μg), at that time and a little later. The amount of radioactive material should be decaying exponentially, so the results of the measurements are shown in the graph.



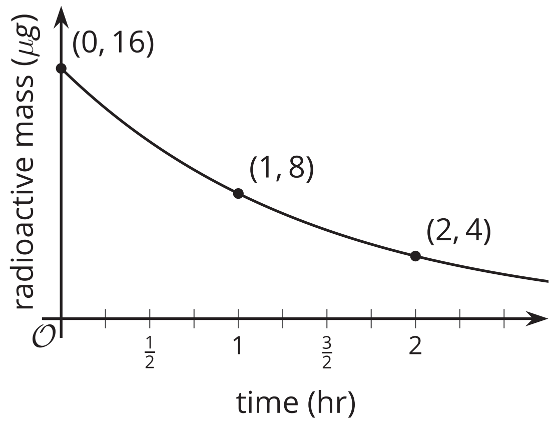
1. After $\frac{1}{4}$ hour there are about 7 μg of material. After $\frac{3}{4}$ hour there are about 3.5 μg of material. About how many μg of material is predicted to be left after $1\frac{3}{4}$ hours? Explain how you know.

2. To be sure of the material that is leaking, scientists want to check the half-life of this material compared to known values. What is the half-life of this material? Explain or show your reasoning.

3. How does the decay rate from $\frac{1}{4}$ hour to $\frac{1}{2}$ hour compare to the decay rate from $\frac{1}{2}$ hour to $\frac{3}{4}$ hour? Explain how you know.

5.4 Average Rate of Exponentials

Here is a graph representing the mass of a radioactive material, in micrograms (μg), as a function of time, in hours, after it was first measured.



1. By what factor does the mass of radioactive material change every hour?

2. For each pair of points, draw a line segment connecting them. Then find the average rate of change. Explain or show your reasoning.
   a. $(0, 16)$ and $(1, 8)$
   b. $(1, 8)$ and $(2, 4)$
   c. $(0, 16)$ and $(2, 4)$

3. Between hours 19 and 20, by what factor do you expect the mass of radioactive material to change? Explain your reasoning.

4. Between hours 19 and 20, what do you expect the average rate of change to be? Be prepared to explain your reasoning.

Lesson 5 Summary

Earlier we learned that, for an exponential function, every time the input increases by a certain amount the output changes by a certain factor.

For example, the population of a country, in millions, can be modeled by the exponential function $f(c) = 5 \cdot 16^c$, where c is time in centuries since 1900. By this model, the growth factor for any one century after the initial measurement is 16.

What about the growth factor for any one decade (one tenth of a century)? Let's start by finding the growth factors between 1910 and 1920 (c between 0.1 and 0.2) and between 1960 and 1970 (c between 0.6 and 0.7). To do that we can calculate the quotients of the function at those input values.

- From 1910 to 1920: $\dfrac{f(0.2)}{f(0.1)} = \dfrac{5 \cdot 16^{(0.2)}}{5 \cdot 16^{(0.1)}}$, which equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)
- From 1960 to 1970: $\dfrac{f(0.7)}{f(0.6)} = \dfrac{5 \cdot 16^{(0.7)}}{5 \cdot 16^{(0.6)}}$, which equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)

Now we can generalize about the growth factor for *any* one decade using the population x centuries after 1900, $f(x)$, and the population one decade (one tenth of a century) after that point, $f(x + 0.1)$.

- From x to $(x + 0.1)$: $\dfrac{f(x+0.1)}{f(x)} = \dfrac{5 \cdot 16^{(x+0.1)}}{5 \cdot 16^x}$, which also equals $16^{(0.1)}$ (or $\sqrt[10]{16}$)

This is consistent with what we know about how exponential functions change over whole-number intervals: They always increase or decrease by equal factors over equal intervals. This is true even when the intervals are fractional.