End Behavior of Rational Functions

Let's explore the end behavior of rational functions.

4.1 Different Divisions, Revisited

Review the completed work of dividing two integers and rewriting the equation. Then, following the same representations shown in the work for the integers, complete all three representations of the polynomial division.

$$\begin{array}{r} 252 \\ 11\,\overline{)\,2775} \\ \underline{2200} \\ 575 \\ \underline{550} \\ 25 \\ \underline{22} \\ 3 \end{array}$$

$$\begin{array}{r} 2x^2 \\ x+1\,\overline{)\,2x^3+7x^2+7x+5} \end{array}$$

$2775 = 11(252) + 3$

$\frac{2775}{11} = 252 + \frac{3}{11}$

$2x^3 + 7x^2 + 7x + 5 =$

$\frac{2x^3 + 7x^2 + 7x + 5}{x+1} =$

4.2 Combined Fuel Economy

In 2000, the Environmental Protection Agency (EPA) reported a combined fuel efficiency for conventional cars that assumes 55% city driving and 45% highway driving. The expression for the combined fuel efficiency of a car that gets x mpg in the city and h mpg on the highway can be written as $\frac{100xh}{55x+45h}$.

1. Several conventional cars have a fuel economy for highway driving that is about 10 mpg higher than for city driving. That is, $h = x + 10$. Write a function f that represents the combined fuel efficiency for cars like these in terms of x.

2. Rewrite f in the form $q(x) + \frac{r(x)}{b(x)}$, where $q(x)$, $r(x)$, and $b(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $b(x)$.

4.3 Exploring End Behavior

function	degree of num.	degree of den.	rewritten in the form of $q(x) + \frac{r(x)}{b(x)}$	end behavior
$g(x) = -\frac{5}{x+2}$				
$h(x) = \frac{7x-5}{x+2}$				
$j(x) = \frac{3x^2+7x-5}{x+2}$				
$k(x) = \frac{2x^3+3x^2+7x-5}{x+2}$				
$m(x) = \frac{x+2}{2x^3+3x^2+7x-5}$				

1. Complete the table to explore the end behavior for rational functions.
2. What do you notice about the end behavior of different types of rational functions?

Are you ready for more?

1. Graph $y = j(x)$ and the line it approaches.

2. Under what conditions would the end behavior of the graph of a rational function approach a line that is not horizontal?

3. Create a rational function that approaches the line $y = 2x - 3$ as x gets larger and larger in either the positive or negative direction.

Lesson 4 Summary

In earlier lessons, we saw rational functions whose end behavior could be described by a horizontal asymptote. For example, we can rewrite functions like $d(x) = \frac{x+4}{x}$ as $d(x) = 1 + \frac{4}{x}$ to see more clearly that as x gets larger and larger in either the positive or negative direction, the value of $\frac{4}{x}$ gets closer and closer to 0, which means the value of $d(x)$ gets closer to 1. We can use similar thinking to understand rational functions that do not have horizontal asymptotes.

For example, consider $f(x) = \frac{x^2+4x+5}{x-3}$. Using division, the expression can be rewritten as $f(x) = x + 7 + \frac{26}{x-3}$. As x gets larger and larger in either the positive or negative direction, the value of the term $\frac{26}{x-3}$ gets closer and closer to 0, which means the value of $f(x)$ gets closer to the value of $x + 7$. This means that the end behavior of f can be described by the line $y = x + 7$. Here is a graph of $y = f(x)$, the diagonal line $y = x + 7$, and the vertical asymptote of the function at $x = 3$: