Unit 8, Lesson 4

Solving Quadratic Equations with the Zero Product Property

Goals

- Given quadratic equations where one side is a product of factors and the other is zero, find the solution(s) and explain (orally and in writing) why the solutions make the equation true.
- Understand that the "zero product property" (in written and spoken language) means that if the product of two numbers is 0, then one of the factors must also be 0.

Learning Targets

- I can explain the meaning of the "zero product property."
- I can find solutions to quadratic equations when one side is a product of factors and the other side is zero.

Lesson Narrative

In this lesson, students learn about the **zero product property**. They use it to reason about the solutions to quadratic equations that each have a quadratic expression in factored form on one side and zero on the other side. They see that when an expression is a product of two or more factors and that product is zero, one of the factors must be zero. This fact enables us to find unknown values in the factored expression.

Students make use of the structure of a quadratic expression in factored form and the zero product property to understand the connections between the numbers in the expression and the x-intercepts of its graph (MP7).

Standards

Building On	**HSA-REI.A.1, HSA-REI.B.3**
Addressing	**HSA-REI.B.4**
Building Toward	**HSA-CED.A.1, HSA-SSE.B.3**

Instructional Routines

- Math Talk
- MLR8: Discussion Supports

Student Facing Learning Goals

Let's find solutions to equations that contain products that equal zero.

4.1 Math Talk: Solve These Equations

Warm-up

10 min

Activity Narrative

This *Math Talk* focuses on introducing the zero product property. It encourages students to think about how to make zero from two factors and to rely on what they know about multiplying to make zero to mentally solve problems. The understanding elicited here will be helpful later in the lesson when students solve quadratic equations in factored form.

To notice the key property, students need to look for and make use of structure (MP7).

Standards

Building On **HSA-REI.A.1**

Instructional Routines

- Math Talk
- MLR8: Discussion Supports

Launch

Tell students to close their books or devices (or to keep them closed). Reveal one problem at a time. For each problem:

- Give students quiet think time, and ask them to give a signal when they have an answer and a strategy.
- Invite students to share their strategies, and record and display their responses for all to see.
- Use the questions in the *Activity Synthesis* to involve more students in the conversation before moving to the next problem.

Access for Students with Disabilities

Action and Expression: Internalize Executive Functions. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory, Organization

Student Task Statement

What values of the variables make each equation true?

- $6 + 2a = 0$
- $7b = 0$
- $7(c - 5) = 0$
- $g \cdot h = 0$

Student Response

- -3. Sample reasoning: To have a sum of 0, the terms must be opposites. That means $2a = -6$ and $a = -3$.
- 0. Sample reasoning: The only number that can be multiplied by 7 to get 0 is 0 itself.
- 5. Sample reasoning: By a similar reasoning, $c - 5 = 0$, so $c = 5$.
- Either $g = 0$ or $h = 0$. Sample reasoning: To multiply and get 0, at least one of the factors must be 0.

Activity Synthesis

To involve more students in the conversation, consider asking:

- "Who can restate ________'s reasoning in a different way?"
- "Did anyone use the same strategy but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to ________'s strategy?"

- "Do you agree or disagree? Why?"
- "What connections to previous problems do you see?"

Highlight explanations that state that any number multiplied by 0 is 0. Then, introduce the **zero product property**, which states that if the product of two numbers is 0, then at least one of the numbers is 0.

Access for English Language Learners

MLR8 Discussion Supports. Display sentence frames to support students when they explain their strategy. For example, "First, I ________ because . . . ." or "I noticed ________, so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Advances: Speaking, Representing

4.2 Take the Zero Product Property Out for a Spin

15 min

Activity Narrative

In this activity, students solve equations of increasing complexity and do so by reasoning. They begin with linear equations, move toward a series of quadratic expressions in factored form, and end with a cubic expression in factored form. The progression prompts students to reason about the parts and structure of the expressions (MP7), rather than to memorize steps for solving without understanding, and to notice regularity through repeated reasoning (MP8).

As students discuss their reasoning with their partner, listen for those who invoke the zero product property to explain how the last four equations could be solved, and for those who notice a pattern in how the equations could be solved.

Standards

Building On	**HSA-REI.A.1, HSA-REI.B.3**
Addressing	**HSA-REI.B.4**

Launch

Arrange students in groups of 2.

Give students 1–2 minutes of independent work time followed by 2–3 minutes to discuss with a partner, then hold a whole-class discussion.

If needed, remind students that some equations have more than one solution. Because we want students to use reasoning and the structure of equations to develop their solutions, discourage use of graphing technology or spreadsheets in this activity.

Student Task Statement

For each equation, find its solution or solutions. Be prepared to explain your reasoning.

1. $x - 3 = 0$
2. $x + 11 = 0$

3. $2x + 11 = 0$
4. $x(2x + 11) = 0$
5. $(x - 3)(x + 11) = 0$
6. $(x - 3)(2x + 11) = 0$
7. $x(x + 3)(3x - 4) = 0$

Student Response

1. 3
2. -11
3. $\frac{-11}{2}$
4. 0 and $\frac{-11}{2}$
5. 3 and -11
6. 3 and $\frac{-11}{2}$
7. 0, -3, and $\frac{4}{3}$

Building on Student Thinking

Students may incorrectly think that x can represent a different value in each factor in an equation. For example, upon finding -11 and 3 as solutions to $(x - 3)(x + 11) = 0$, they think that one solution is for the x in $(x - 3)$ and the other for the x in $(x - 11)$.

Remind students that solving the equation $(x - 3)(x + 11) = 0$ is like finding the zeros of the function defined by $(x - 3)(x - 11)$. Although there may be two values of x that lead to 0 for the value of$(x - 3)(x - 11)$, only one input can be entered into the function at a time. Ask students to substitute the solutions into the equations and check if the expression is equal to 0 each time.

- When $x = \text{-}11$, the value of the expression is $(\text{-}11 - 3)(\text{-}11 + 11)$, or $(\text{-}14)(0)$, which is 0.
- When $x = 3$, the value of the expression is $(3 - 3)(3 + 11)$, or $(0)(14)$, which is 0.

Are You Ready for More?

1. Use factors of 48 to find as many solutions as you can to the equation $(x - 3)(x + 5) = 48$.
2. Once you think you have all the solutions, explain why these must be the only solutions.

Extension Student Response

1. 7 and -9
2. Sample response: The numbers expressed by $x - 3$ and $x + 5$ are 8 units apart. The only factor pairs of 48 that are 8 units apart are 4 and 12 and -4 and -12. If $x = 7$, then $(x - 3)(x + 5) = 4 \cdot 12$. If $x = \text{-}9$, then $(x - 3)(x + 5) = \text{-}12 \cdot (\text{-}4)$.

Activity Synthesis

Invite students to share their strategies for solving the nonlinear equations. As they explain, record and organize each step of their reasoning process, and display for all to see.

For example, the equation $(x-3)(2x+11)=0$ tells us that if the product of $(x-3)$ and $(2x+11)$ is 0, then either $x-3$ is equal to 0 or $2x+11$ is equal to 0. We can then organize the rest of the solving process as:

If $x-3$ is equal to 0, then x is 3.

$$\begin{aligned} x-3&=0\\ x&=3 \end{aligned}$$

If $2x+11$ is equal to 0, then $x=-\frac{11}{2}$

$$\begin{aligned} 2x+11&=0\\ 2x&=-11\\ x&=-\frac{11}{2} \end{aligned}$$

The equation is true when $x=3$ or when $x=-\frac{11}{2}$.

Emphasize that because at least one of the factors must be 0 for the product to be 0, we can set each expression that is a factor equal to 0 and solve each of these equations separately.

Remind students that we can check our solutions by substituting each one back into the equation and seeing if the equation remains true. Although the two factors, $(x-3)$ and $(2x+11)$, won't be 0 simultaneously when 3 or $-\frac{11}{2}$ is substituted for x, the expression on the left side of the equation will have a value of 0 because one of the factors is 0.

- When x is 3, the expression is $(3-3)\,(2(3)+11)$, or $(0)(17)$, which is 0.
- When x is $-\frac{11}{2}$, the expression is $\left(-\frac{11}{2}-3\right)\left(2(-\frac{11}{2})+11\right)$, or $\left(-\frac{17}{2}\right)(0)$, which is 0.

4.3 Revisiting a Projectile

10 min

Activity Narrative

This activity enables students to apply the zero product property to solve a contextual problem and reinforces the idea of solving quadratic equations as a way to reason about quadratic functions.

Previously, students have encountered two equivalent quadratic expressions that define the same quadratic function. Here, they work to show that two quadratic expressions—one in standard form and the other in factored form—really do define the same function.

Monitor for these likely strategies, and select students who use various strategies to share in the discussion. Students may:

- Graph both equations on the same coordinate plane and show that they coincide.
- Inspect a table of values of both equations and show that the same output results for any input.
- Use the distributive property to multiply the expression in factored form to show that $(-5t-3)(t-6)=-5t^2+27t+18$.

Next, they consider whether the standard form or factored form better helps them find the zeros of the function. They then use that form to find the zeros without graphing. The work here reiterates the connections between finding the zeros of a quadratic function and solving a quadratic equation where a quadratic expression that defines a function has a value of zero.

Standards

Addressing **HSA-REI.B.4**

Instructional Routines

- MLR8: Discussion Supports

Launch

Keep students in groups of 2. Prepare access to graphing technology and spreadsheet tool, if requested.

Display the two equations that define h for all to see. Tell students that the two equations define the same function. Ask students how they could show that the two equations indeed define the same function.

Give students a moment of quiet time to think of a strategy and test it, then time to discuss with a partner. Then, discuss their responses.

Once students see some evidence, ask students to proceed to the activity.

Student Task Statement

We have seen quadratic functions modeling the height of a projectile as a function of time.

Here are two ways to define the same function that approximates the height of a projectile in meters, t seconds after launch:

$$h(t) = -5t^2 + 27t + 18 \qquad h(t) = (-5t - 3)(t - 6)$$

1. Which way of defining the function allows us to use the zero product property to find out when the height of the object is 0 meters?
2. Without graphing, determine at what time the height of the object is 0 meters. Show your reasoning.

Student Response

1. the equation with a quadratic expression in factored form
2. The object has a height of 0 meters after 6 seconds. Sample reasoning: Applying the zero product property to solve $(-5t - 3)(t - 6) = 0$ gives $t = -\frac{3}{5}$ and $t = 6$. Because the object was launched at $t = 0$, the negative solution doesn't make sense in this context.

Activity Synthesis

Ask students to share their responses and reasoning. Discuss questions such as:

- "Why is the factored form more helpful for finding the time when the object has a height of 0 meters?" (To find the input values when the output has a value of 0 is to solve the equation $\text{quadratic expression} = 0$. When the expression is in factored form, we can use the zero product property to find the unknown inputs.)
- "What if we tried to solve the equation in standard form by performing the same operation on each side?" (We would get stuck. For instance, we could add or subtract terms from each side, but then there are no like terms to combine on either side, so we are no closer to isolating the variable.)

If no students related solving equations in factored form to using the factored form to find the horizontal intercepts of a graph of a quadratic function, discuss this connection.

- "In an earlier unit, we saw that the factored form of a quadratic expression, such as $(x - 5)(x + 9)$, allows us to see the x-intercepts of its graph, but we didn't look into why the graph crosses the x-axis at those points. Can you explain why it does now?" (The x-intercepts have a y-value of 0, which means the quadratic function is 0 at those x-values: $(x - 5)(x + 9) = 0$. If multiplying two numbers gives 0, one of them must be 0. So either $x - 5 = 0$ or $x + 9 = 0$. If $x - 5 = 0$, then x is 5. If $x + 9 = 0$, then x is -9.)

Access for English Language Learners

MLR8 Discussion Supports. Revoice student ideas to demonstrate and amplify mathematical language use. For example, revoice the student statement "The first one is more helpful" as a question such as "Do you agree that the factored form is more helpful for finding when the object has a height of 0 meters?"
Advances: Speaking, Listening

Lesson Synthesis

To help students consolidate the ideas in the lesson, discuss questions such as:

- "How does the **zero product property** help us find the solutions to $(x-3)(x+4)=0$?" (It tells us that either $x-3=0$ or $x+4=0$," and each of these equations can be solved easily.)
- "Can you explain why the solutions to $(x-3)(x+4)=8$ are *not* 3 and -4?" (The zero product property works only when the product of the factors is 0. When the product is any other number, we can't conclude that each factor is that number.)
- "The expression x^2-x-12 is equivalent to $(x+3)(x-4)$. Can we apply the zero product property to solve $x^2-x-12=0$?" (Only if we rewrite the expression on the left in factored form first. We can't use the zero product property when the expression is not a product of factors.)
- "Can we solve $x^2-x-12=0$ by performing the same operation to each side of the equation?" (No, doing that doesn't help us isolate the variable.)

4.4 Solve This Equation!

Cool-down

5 min

Standards

Addressing **HSA-REI.B.4**

Launch

No technology should be used for this *Cool-down*.

Student Task Statement

Find all solutions to $(x+5)(2x-3)=0$. Explain or show your reasoning.

Student Response

-5 and $\frac{3}{2}$. Sample reasoning: By the zero product property, either $x+5=0$ or $2x-3=0$, so $x=$ -5 or $x=\frac{3}{2}$.

Responding to Student Thinking

More Chances

Students will have more opportunities to understand the mathematical ideas addressed here. There is no need to slow down or add additional work to the next lessons.

Lesson 4 Summary

The **zero product property** says that if the product of two numbers is 0, then one of the numbers must be 0. In other words, if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. This property is handy when an equation we want to solve states that the product of two factors is 0.

Suppose we want to solve $m(m + 9) = 0$. This equation says that the product of m and $(m + 9)$ is 0. For this to be true, either $m = 0$ or $m + 9 = 0$, so both 0 and -9 are solutions.

Here is another equation: $(u - 2.345)(14u + 2) = 0$. The equation says the product of $(u - 2.345)$ and $(14u + 2)$ is 0, so we can use the zero product property to help us find the values of u. For the equation to be true, one of the factors must be 0.

- For $u - 2.345 = 0$ to be true, u would have to be 2.345.
- For $14u + 2 = 0$ or ($14u =$ -2) to be true, u would have to be $-\frac{2}{14}$, or $-\frac{1}{7}$.

The solutions are 2.345 and $-\frac{1}{7}$.

In general, when a quadratic expression in factored form is on one side of an equation and 0 is on the other side, we can use the zero product property to find its solutions.

This property is unique to 0. Given an equation like $a \cdot b = 6$, the factors could be 2 and 3, 1 and 6, -12 and $-\frac{1}{2}$, π and $\frac{6}{\pi}$, or any other of the infinite number of combinations. This type of equation does not give insight into the value of a or b.

Glossary

- zero product property

Lesson 4 Practice Problems

1 Student Task Statement

If the equation $(x + 10)x = 0$ is true, which statement is also true according to the zero product property?

A. Only $x = 0$.

B. Either $x = 0$ or $x + 10 = 0$.

C. Either $x^2 = 0$ or $10x = 0$.

D. Only $x + 10 = 0$.

Solution

B

2 Student Task Statement

What are the solutions to the equation $(10 - x)(3x - 9) = 0$?

A. -10 and 3

B. -10 and 9

C. 10 and 3

D. 10 and 9

Solution

C

3 Student Task Statement

Solve each equation.

a. $(x - 6)(x + 5) = 0$

b. $(x - 3)(\frac{2}{3}x - 6) = 0$

c. $(-3x - 15)(x + 7) = 0$

Solution

a. $x = -5$ and $x = 6$

b. $x = 3$ and $x = 9$

c. $x = -5$ and $x = -7$

4 **Student Task Statement**

Consider the expressions $(x-4)(3x-6)$ and $3x^2 - 18x + 24$.

Show that the two expressions define the same function.

Solution

Students may multiply the factors in the first expression to show that it is algebraically equivalent to the second, or they may graph both functions and show they produce the same graph. (Only the former proves that the two expressions are equivalent, but the latter is sufficient for informally showing that they are the same function.)

5 **Student Task Statement**

Kiran saw that if the equation $(x+2)(x-4) = 0$ is true, then by the zero product property, either $x+2$ is 0 or $x-4$ is 0. He then reasoned that, if $(x+2)(x-4) = 72$ is true, then either $x+2$ is equal to 72 or $x-4$ is equal to 72.

Explain why Kiran's conclusion is incorrect.

Solution

Sample response: The zero product property applies only to products that are equal to 0, not any product. Many pairs of factors can be multiplied to be 72. We can't be sure that one of the factors is equal to 72.

6 from Unit 8, Lesson 2

Student Task Statement

Andre wants to solve the equation $5x^2 - 4x - 18 = 20$. He uses a graphing calculator to graph $y = 5x^2 - 4x - 18$ and $y = 20$ and finds that the graphs cross at the points $(-2.39, 20)$ and $(3.19, 20)$.

a. Substitute each x-value that Andre found into the expression $5x^2 - 4x - 18$. Then evaluate the expression.
b. Why did neither solution make $5x^2 - 4x - 18$ equal exactly 20?

Solution

a. Both x-values result in 20.1205, not 20, for the value of the expression.
b. Graphing software gives solutions that have been rounded. To know the exact solutions, we need to solve the equation without graphing.

7 from Unit 8, Lesson 3

Student Task Statement

Select **all** the solutions to the equation $7x^2 = 343$.

A. 49

B. $-\sqrt{7}$

C. 7

D. -7

E. $\sqrt{49}$

F. $\sqrt{-49}$

G. $-\sqrt{49}$

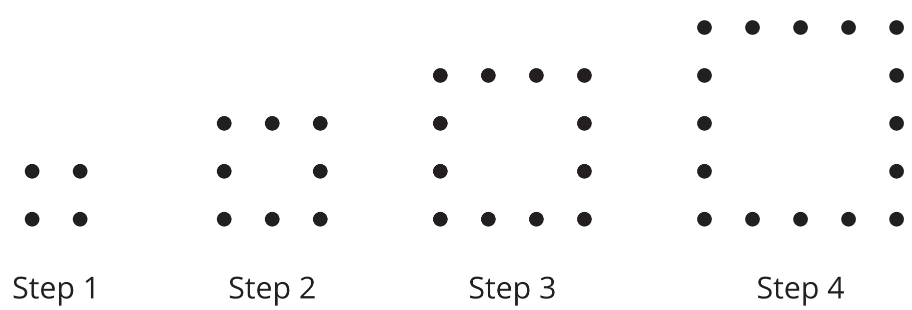
Solution

C, D, E, G

8 from Unit 7, Lesson 2

Student Task Statement

Han says this pattern of dots can be represented by a quadratic relationship because the dots are arranged in a square in each step.



Do you agree? Explain your reasoning.

Solution

I disagree. This is a linear relationship. The total number of dots is growing by 4 for each new step.